INDEFINITELY DIFFERENTIABLE FUNCTIONS WITH PRESCRIBED LEAST UPPER BOUNDS

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1. Introduction. Let \( F(x) \) be a real indefinitely differentiable function of the real variable \( x \) defined on the interval \( a \leq x \leq b \), and let \( M_n \) denote the least upper bound of \( |F^{(n)}(x)| \) on that interval. In this paper we shall establish sufficient conditions that there exist an indefinitely differentiable function \( F(x) \) taking on certain prescribed \( M_n \).

It is easy to see that \( M_0 \) and \( M_1 \) can be assigned arbitrarily. However, the first three upper bounds \( M_0, M_1, \) and \( M_2 \) must satisfy certain inequalities.

Let us consider the interval \((0, 1)\). Let \( t_1 \) be the value of \( x \) for which \( |F^{(1)}(x)| \) attains its maximum. Then

\[
F(1) - F(t_1) = (1 - t_1)F^{(1)}(t_1) + (1/2!)F^{(2)}(t_1)(1 - t_1)^2,
\]

where \( t_1 < \theta_1 < 1 \). And similarly

\[
F(0) - F(t_1) = -t_1F^{(1)}(t_1) + (1/2!)F^{(2)}(t_1)t_1^2,
\]

where \( 0 < \theta_2 < t_1 \). On subtracting these equations we obtain

\[
F^{(1)}(t_1) = F(1) - F(0) + (1/2!)\left\{F^{(2)}(\theta_2)t_1^2 - F^{(2)}(\theta_1)(1 - t_1)^2\right\},
\]

\[
M_1 \leq 2M_0 + M_2/2!.
\]

By the same procedure we can obtain for the interval \((0, a)\)

\[
M_1 \leq 2M_0/a + M_2a/2!.
\]

In the case of the interval \((0, \infty)\) we can replace \((1)\) by a more precise inequality. Since \( a \) is arbitrary, we can replace \( a \) by the positive value which minimizes the right side of \((1)\), and obtain

\[
M_1 \leq 2(M_0M_2)^{1/2}.
\]

Ore\(^4\) in a recent paper employed the results of W. Markoff to obtain certain inequalities connecting the least upper bounds of \( |F^{(i)}(x)|, (1 \leq i \leq n) \), with those of \( |F(x)| \) and \( |F^{(n+1)}(x)| \) where \( F(x) \) is a function with bounded \((n+1)\)th derivative. For the first derivative the inequality \((1)\) is slightly better than that obtained by Ore.

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2 By \( F^{(0)}(x) \) we shall mean \( F(x) \).

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2. Construction of an indefinitely differentiable function with prescribed least upper bounds. We now prove the following theorem.

**Theorem.** If the sequence \( \{ M_i a^i \} \) is monotone decreasing, then there exists an indefinitely differentiable function \( F(x) \) defined on \( (0 \leq x \leq a) \) such that \( M_i \) is the least upper bound of \( |F^{(i)}(x)| \) on \( (0 \leq x \leq a) \).

Define

\[
0 \leq S_i = \sum_{j=0}^{\infty} (-1)^j \frac{M_{i+j} a^j}{j!} \leq M_i, \quad F(x) = \sum_{i=0}^{\infty} S_i \frac{x^i}{i!}.
\]

Now the function \( F(x) \) so defined is an entire function. Let \( x = b \). Then

\[
F(b) = \sum_{i=0}^{\infty} S_i \frac{b^i}{i!} = \sum_{i=0}^{\infty} (-1)^i \frac{M_i}{i!} \left( \sum_{j=0}^{i} (-1)^j C_{i-j,j} a^{i-j} b^j \right),
\]

and since the series

\[
\sum_{i=0}^{\infty} M_i a^i \left(1 - b/a\right)^i \leq M_0 \sum_{i=0}^{\infty} \frac{\left(1 - b/a\right)^i}{i!}
\]

converges, the series \( \sum_{i=0}^{\infty} (S_i/i!) b^i \) converges. Further,

\[
F^{(i)}(a) = \sum_{j=0}^{\infty} (-1)^j M_{i+j} a^j \left( \sum_{k=0}^{j} \frac{(-1)^k}{k!(j-k)!} \right) = M_i,
\]

and

\[
|F^{(i)}(x)| \leq F^{(i)}(a) = M_i, \quad 0 \leq x \leq a.
\]

Thus we have given explicitly a function \( F(x) \) satisfying the conditions of the theorem.

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