A DECOMPOSITION OF ADDITIVE SET FUNCTIONS

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This paper is concerned with a decomposition theorem for additive functions on an additive family of sets to either real numbers or a Banach space. Additive bounded set functions have as yet been little studied. However the recent paper of Hildebrandt illustrates their importance.

We shall use the following notation:

(a) \( T \): an abstract class of arbitrary elements.

(b) \( \mathcal{J} \): a completely additive family of subsets \( T \) of \( T \); that is, \( T \in \mathcal{J} \), \( \tau \in \mathcal{J} \) implies \( T - \tau \in \mathcal{J} \), and \( \tau_n \in \mathcal{J} \) for \( n = 1, 2, \ldots \) implies \( \sum \tau_n \in \mathcal{J} \).

(c) \( \alpha \): a set function on \( \mathcal{J} \) to real numbers.

(d) \( \mathcal{A} \): the subclass of set functions on \( \mathcal{J} \) to real numbers which are additive and bounded; that is, \( \tau_1, \tau_2 \in \mathcal{J} \) and \( \tau_1 \cdot \tau_2 = 0 \) implies \( \alpha(\tau_1 + \tau_2) = \alpha(\tau_1) + \alpha(\tau_2) \).

(e) \( \mathcal{C} \): the subclass of set functions on \( \mathcal{J} \) to real numbers which are completely additive (c.a.), that is, \( \tau_n \in \mathcal{J} \) for \( n = 1, 2, \ldots \) and \( \tau_i \cdot \tau_j = 0 \) if \( i \neq j \) implies \( \alpha(\sum \tau_n) = \sum \alpha(\tau_n) \). The functions in \( \mathcal{C} \) are bounded.\(^3\)

The notations \( \mathcal{A}_P \) and \( \mathcal{C}_P \) refer to the subclasses of \( \mathcal{A} \) and \( \mathcal{C} \) respectively whose elements are nonnegative.

(f) \( x \): a set function on \( \mathcal{J} \) to a Banach space \( X \). The definitions of additive and c.a. set functions are formally retained. If \( \{ \tau_n \} \) is a sequence of disjoint sets of \( \mathcal{J} \) and \( x(\tau) \) is c.a., then \( \sum x(\tau_n) \) is unconditionally convergent.\(^5\)

(g) \( \mathcal{C}_X \): the class of c.a. set functions on \( \mathcal{J} \) to \( X \).

In the statement of the following theorems, \( \mathcal{D} \) will designate any one of the classes \( \mathcal{A}, \mathcal{A}_P, \mathcal{C}, \mathcal{C}_P \), and \( \mathcal{T} \) will denote the cardinal number of \( \tau \).

**Theorem 1.** Let \( \aleph \) be an infinite cardinal number not greater than \( \overline{\mathcal{T}} \). For every \( \alpha \in \mathcal{D} \) there exists an unique decomposition \( \alpha = \alpha_1 + \alpha_2 \) and a set \( R(\alpha) \in \mathcal{J} \) of cardinal number not greater than \( \aleph \) such that \( \alpha_1, \alpha_2 \in \mathcal{D} \).

\(^1\) Presented to the Society April 15, 1939, under the title On additive set functions.


\(^4\) S. Banach, Théorie des Opérations Linéaires, Monografje Matematyczne, Warsaw, 1932, chap. 5.

\(^5\) If \( x_n \) is a series of elements of \( X \) and if every subseries \( \sum x_n \) is convergent, then \( \sum x_n \) is said to be unconditionally convergent.
ADDITIVE SET FUNCTIONS

Let $\alpha_1(\tau) = \alpha(\tau \cdot T)$, $\alpha_2(\tau) = 0$ if $\tau \leq \aleph_0$.

Let $\Sigma = E_r [r \in \mathcal{R}, \tau \leq \aleph_0, \alpha(\tau) \neq 0]$. We define a transfinite sequence $(\tau_1, \tau_2, \ldots; \tau_n, \ldots, \tau_m, \ldots)$ as follows: $\tau_1$ is an arbitrary element of $\Sigma$. Suppose $\tau_k$ have been defined for all $\lambda < \mu$. If there exists $\tau$ such that $\tau \cdot \sum_{\lambda < \mu} \alpha_\lambda = 0$ and $\tau \in \Sigma$, then we set $\tau = \tau_\mu$.

As $\alpha(\tau)$ is bounded, $\alpha(\tau)$ cannot differ from zero on a nondenumerable number of disjoint sets. The sequence therefore contains only a denumerable set of elements.

Let $\mathcal{R} = \sum_{\lambda \tau \lambda}$. Then $\mathcal{R} \in \mathfrak{S}$ and $\mathcal{R} \leq \aleph_0$. We define $\alpha_1(\tau) = \alpha(\mathcal{R} \cdot \tau)$, $\alpha_2(\tau) = \alpha(\tau) - \alpha_1(\tau) = \alpha(\tau - \mathcal{R} \cdot \tau)$. The $\alpha_1(\tau)$, $\alpha_2(\tau)$ are clearly elements of $\mathfrak{S}$. If $\tau \leq \aleph_0$, then by the definition of $\mathcal{R}$, $\alpha_1(\tau) = 0$.

Although the set $\mathcal{R}$ is not unique, the function decomposition is unique: Suppose there exist two different sets $\mathcal{R}_1$, $\mathcal{R}_2$ having the properties of the $\mathcal{R}$ defined above. The set identity $\mathcal{R}_1 \cdot \tau + (\mathcal{R}_2 - \mathcal{R}_1) \cdot \tau = \mathcal{R}_1 \cdot \tau + (\mathcal{R}_2 - \mathcal{R}_1) \cdot \tau$ and $\alpha[(\mathcal{R}_1 - \mathcal{R}_2) \cdot \tau] = 0 = \alpha[(\mathcal{R}_2 - \mathcal{R}_1) \cdot \tau]$ imply that $\alpha(\mathcal{R}_1 \cdot \tau) = \alpha(\mathcal{R}_2 \cdot \tau)$.

A set function $\alpha$ on $\mathfrak{S}$ will be said to be nonsingular if for every $t \in \mathfrak{S}$, $\alpha(t) = 0$. A set function $\alpha$ on $\mathfrak{S}$ will be called $\aleph_0$-homogeneous if there exists a set $\mathcal{R}$ such that $\mathcal{R} \in \mathfrak{S}$, $\mathcal{R} \leq \aleph_0$, $\alpha(\tau) = \alpha(\mathcal{R} \cdot \tau)$, and $\alpha(\tau) = 0$ if $\tau < \aleph_0$.

Without loss of generality we may consider only nonsingular set functions because for every $\alpha \in \mathfrak{S}$ there exists a unique decomposition $\alpha = \alpha_1 + \alpha_2$ and a denumerable set $\{t_i\}$ of elements of $\mathcal{T}$, such that $\alpha_1$, $\alpha_2 \in \mathfrak{S}$, $\alpha_1(\tau) = \sum_{i=0}^{\infty} \alpha_1(\tau \cdot t_i)$, and $\alpha_2$ is nonsingular. We omit the proof.

**Theorem 2.** For every nonsingular $\alpha \in \mathfrak{S}$, there exists an unique decomposition $\alpha = \sum \alpha_i$, the sum being absolutely convergent, and such that $\alpha_i$ is $\aleph_0$-homogeneous and $\aleph_0 \neq \aleph_j$; if $i \neq j$.

In the proof of this theorem an induction is made on the infinite cardinals not exceeding that of $\mathcal{T}$, well-ordered according to magnitude. We define a transfinite sequence of set functions $(\alpha_1, \alpha_2, \ldots; \alpha_n, \ldots, \alpha_\lambda, \ldots)$ as follows: Suppose $\alpha_\lambda$ have been defined for all $\lambda < \mu$ and (1) only a denumerable number of the $\alpha_\lambda$ are not identically zero; (2) $\sum_{\lambda \leq \mu} |\alpha_\lambda(\tau)| < \infty$; and (3) $\alpha_\lambda \in \mathfrak{S}$ and is $\aleph_0$-homogeneous. By Theorem 1 there exist $\mathcal{R}_\mu \in \mathfrak{S}$ and a decomposition $\alpha = \alpha_1^1 + \alpha_2^2$ such that $\mathcal{R}_\mu \leq \aleph_0$, $\alpha_1^1(\tau) = \alpha(\mathcal{R}_\mu \cdot \tau)$, $\alpha_2^1(\tau) = 0$ if $\tau \leq \aleph_0$, and $\alpha_1^2, \alpha_2^2 \in \mathfrak{S}$. Clearly $\alpha_i(\tau) = \alpha(\mathcal{R}_\mu \cdot \mathcal{R}_i \cdot \tau)$ if $\lambda < \mu$.

Let $\alpha_\mu(\tau) = \alpha_\mu(\tau) - \sum_{\lambda < \mu} \alpha_\lambda(\tau)$. We consider the following cases:

1. $\alpha \in \mathcal{C}$, $\mathcal{C}_p$. Let $\mathcal{R}_\mu = \mathcal{R}_\mu - \sum_{t \in \mathfrak{S}} \mathcal{R}_\lambda$ where $\pi_\mu = E_\lambda[\lambda < \mu, \alpha_\lambda \neq 0]$. The sets $\mathcal{R}_\mu$ are disjoint. Suppose $\alpha_\lambda(\tau) = \alpha(\mathcal{R}_\lambda \cdot \tau)$ for $\lambda < \mu$. Then by (1)
\[ \alpha_\mu(\tau) = \alpha(R_\mu \cdot \tau) - \sum_{\pi_\mu} \alpha_\lambda(\tau) = \alpha(R_\mu \cdot \tau) - \sum_{\pi_\mu} \alpha(R_\mu \cdot \overline{R}_\lambda \cdot \tau) \]

\[ = \alpha \left[ \left( R_\mu - \sum_{\pi_\mu} R_\mu \cdot \overline{R}_\lambda \right) \cdot \tau \right] = \alpha(R_\mu \cdot \tau). \]

It is clear that (1), (2), and (3) are satisfied for \( \mu + 1 \). \( \alpha_\lambda \neq 0 \) implies that \( \alpha(\tau) \neq 0 \) for some subset of \( \overline{R}_\lambda \). As the \( \overline{R}_\lambda \) are disjoint, the sequence will contain only a denumerable number of functions not identically zero.

II. \( \alpha \in A_P \). For \( \lambda_0 < \mu, \alpha(T) \geq a^1_\lambda(T) = \sum_{\lambda \leq \lambda_0} \alpha(T) = \sum_{\lambda \leq \lambda_0} \alpha_\lambda(T) \). Clearly (1) and (2) are satisfied for \( \mu + 1 \), and the sequence contains only a denumerable number of functions not identically zero. Let \( \lambda_i \) be a spanning sequence for \( E_\lambda[\lambda < \mu, \alpha_\lambda \neq 0] \). Then

\[ \alpha_\mu(\tau) = \alpha_\mu^1(\tau) - \sum_{\lambda < \mu} \alpha_\lambda(\tau) = \alpha(R_\mu \cdot \tau) - \lim_{\tau \to \infty} \alpha_\mu^1(\tau) \]

\[ = \alpha(R_\mu \cdot \tau) - \lim_{\tau \to \infty} \alpha(R_\mu \cdot R_{\lambda_i} \cdot \tau). \]

Hence (3) is likewise satisfied.

III. \( \alpha \in A \). Every \( \alpha \in A \) has a decomposition \( \alpha = \alpha_1 - \alpha_2 \) where \( \alpha_1, \alpha_2 \in A_P \). An application of II to \( \alpha_1 \) and \( \alpha_2 \) gives the desired decomposition.

The decomposition is unique: Any two sequences of homogeneous functions differ in a first function, \( \alpha_\mu \). But this is contrary to \( \alpha^1_\mu = \sum_{\lambda \leq \mu} \alpha_\lambda \) being unique.

In these theorems the restriction that the additive bounded set function be defined over an additive family \( \mathcal{S} \) is optional, since the range of definition of such a function can always be extended to an additive family. The type of argument used by Pettis\(^6\) will prove this statement.

We next consider the possibility of extending these theorems to functions \( x(\tau) \) on \( \mathcal{S} \) to a Banach space. The theorem is not in general valid for additive bounded set functions of this type. This is illustrated by \( x(\tau) \) defined on all subsets of \( T = (0, 1) \) to the space \( X \) of bounded functions on \( S = (0, 1) \) where \( x(\tau) \) is the characteristic function of the subset of \( S \) which has the same coordinate values as \( \tau \). Clearly there exists no denumerable set \( R \) such that \( x(\tau - R\tau) = 0 \) for all denumerable sets \( \tau \).

However analogous theorems are obtained for c.a. set functions on \( \mathcal{S} \) to \( X \).

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Theorem 3. Let $\mathfrak{N}$ be an infinite cardinal number not greater than $\mathfrak{T}$. For every $x \in C_X$ there exists a unique decomposition $x = x_1 + x_2$ and a set $R(x) \in \mathcal{S}$ of cardinal power not greater than $\mathfrak{N}$ such that $x_1, x_2 \in C_X$, $x_1(\tau) = x(R_1 \cdot \tau)$, $x_2(\tau) = 0$ if $\tau \not\leq \mathfrak{N}$.

$x(\tau) \not= 0$ on at most a denumerable number of disjoint sets of $\mathcal{S}$. Suppose the contrary. Then there exists a denumerable sequence of disjoint sets $\{\tau_i\}$ and an $e > 0$ such that $\|x(\tau_i)\| > e$, $(i = 1, 2, \cdots)$. As $x(\tau)$ is c.a., $\sum_i x(\tau_i)$ converges. The supposition is therefore false.

The argument used in Theorem 1 will now prove the theorem.

Theorem 4. For every nonsingular $x \in C_X$, there exists an unique decomposition $x = \sum x_i$, the sum being unconditionally convergent, and such that $x_i$ is $\mathfrak{N}_i$-homogeneous and $\mathfrak{N}_i \neq \mathfrak{N}_j$ if $i \neq j$.

The proof is identical with that of I in Theorem 2. Again there will exist disjoint $R_\mu$'s such that $x_\mu(\tau) = x(R_\mu \cdot \tau)$.

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