A REMARK ON THE SUM AND THE INTERSECTION OF TWO NORMAL IDEALS IN AN ALGEBRA

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Let $F$ be a quotient field of a commutative domain of integrity $o$ in which the usual arithmetic holds. Consider an algebra $\mathfrak{A}$ with a unit element over $F$. Let $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \mathfrak{J}_4$ be four arbitrary maximal orders in $\mathfrak{A}$ and $a, b, c$ be three arbitrary normal ideals. We prove the following theorems.

**Theorem 1.** If $\mathfrak{J}_1 \cap \mathfrak{J}_2 = \mathfrak{J}_3 \cap \mathfrak{J}_4$ [or $(\mathfrak{J}_1, \mathfrak{J}_2) = (\mathfrak{J}_3, \mathfrak{J}_4)$], then either $\mathfrak{J}_1 = \mathfrak{J}_3$, $\mathfrak{J}_2 = \mathfrak{J}_4$ or $\mathfrak{J}_1 = \mathfrak{J}_4$, $\mathfrak{J}_2 = \mathfrak{J}_3$.

**Theorem 2.** Both the left and the right orders of $(\mathfrak{J}_1, \mathfrak{J}_2)$ are $\mathfrak{J}_1 \cap \mathfrak{J}_2$. Also $\mathfrak{J}_1 \cap \mathfrak{J}_2 \subseteq \mathfrak{J}_3$ if and only if $(\mathfrak{J}_1, \mathfrak{J}_2) \supseteq \mathfrak{J}_3$; if this is the case the distance ideal $\mathfrak{d}_{31}$ of $\mathfrak{J}_3$ to $\mathfrak{J}_1$ is divisible by the distance ideal $\mathfrak{d}_{31}$ of $\mathfrak{J}_3$ to $\mathfrak{J}_1$.

**Theorem 3.** The left, say, order $\mathfrak{o}$ of the intersection $a \cap b \ [\text{the sum } (a, b)]$ is an intersection of two suitable maximal orders.

More precisely, if $\mathfrak{r}$ and $\mathfrak{s}$ are normal ideals such that $b = \mathfrak{r} \mathfrak{a} \mathfrak{s}$ in the sense of proper multiplication and if $t$ is the smallest two-sided ideal of the right order of $\mathfrak{a}$ which divides $\mathfrak{s}$ while $t'$ is the largest two-sided ideal of the same maximal order which is divisible by $\mathfrak{s}$, then $\mathfrak{o}$ is the intersection of the left orders of the two normal ideals $a \cap \mathfrak{r} \mathfrak{a} \mathfrak{s}$ and $a \cap \mathfrak{r} \mathfrak{a} \mathfrak{s}' \ [(a, \mathfrak{r} \mathfrak{a} \mathfrak{s}) \text{ and } (a, \mathfrak{r} \mathfrak{a} \mathfrak{s}')]$. The left order of $a \cap b$ coincides with the right order of $(a^{-1}, b^{-1})$.

**Theorem 4.** $a \cap b \subseteq c$ implies $(a^{-1}, b^{-1}) \supseteq c^{-1}$ and conversely.

For the proof we have, according to the well known reduction, only to treat the case where $F$ is a $p$-adic field $F = F_p$ and $\mathfrak{A}$ is a normal simple algebra over $F$. Then $\mathfrak{A}$ is a (complete) matric ring $D_r = \sum_{i,k=1}^r \epsilon_{ik} D$ over a division algebra $D$, where $\epsilon_{ik}$ is a system of matric units commutative with every element of $D$. $D$ possesses a unique maximal order $I$, and $I$ has a unique prime ideal $P$.

**Notation.** If $a_{ik}, \ (i, k = 1, 2, \cdots, r)$, is a system of rational integers, we denote by $M(a_{ik})$ the ideal $\sum_{i,k} \epsilon_{ik} P_{\alpha_{ik}}$ in $\mathfrak{A}$.

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1 In the following we shall adopt the terminologies used in M. Deuring, Algebren, Ergebnisse der Mathematik, vol. 4, no. 1, 1935.
2 If the algebra is a quaternion algebra, then the converse is also valid. Cf. M. Eichler, Journal für die reine und angewandte Mathematik, vol. 174 (1936), §7.
3 Thus the intersection and the sum are no more normal ideals except for trivial cases; cf. Nakayama, Proceedings of the Imperial Academy of Japan, vol. 12 (1936).
$M(a_{ik})$ is an order if and only if $a_{ii} = 0$, $a_{ij} + a_{ik} \geq a_{ik}$ for all $i, k, l$. On assuming this condition it is a maximal order if and only if $\sum a_{ik} = 0$. By a simple calculation we then have the following lemma.4

**Lemma 1.** A necessary and sufficient condition that $M(a_{ik})$ be a maximal order is that there should exist $r$ rational integers $c_i$ such that $a_{ik} = c_k - c_i$. Every normal ideal whose left and right orders are $M(c_k - c_i)$ and $M(d_k - d_i)$ respectively has the form $P^a M(d_k - c_i) = M(d_k - c_i + a)$.

It follows from a lemma of Chevalley6 that a maximal order in $A$ has really the form $M(a_{ik})$ (whence the form $M(c_k - c_i)$) whenever it contains all diagonal $e_1, e_2, \ldots, e_r$.

**Lemma 2.** There exists a regular element $\alpha$ in $A$ such that

$$\alpha^{-1} \mathfrak{I}_1 \alpha = M(0), \quad \alpha^{-1} \mathfrak{I}_2 \alpha = M(c_k - c_i); \quad c_1 \geq c_2 \geq \cdots \geq c_r.$$

**Proof.** There is, as is well known, a regular element $\beta$ such that $\beta^{-1} \mathfrak{I}_1 \beta = M(0)$. Consider the distance ideal $\mathfrak{I}_2 = (\mathfrak{I}_2 \mathfrak{I}_1)^{-1} = \mathfrak{I}_1 \mathfrak{S}$ of $\mathfrak{I}_1$ to $\mathfrak{I}_2$. The theory of elementary divisors tells the existence of two units $\xi, \eta$ in $M(0)$ such that $\gamma = \xi \beta^{-1} \delta \beta \eta = \sum \epsilon_i P^c_i$, where we denote, for the sake of convenience, a prime element of the prime ideal $P$ by the same letter $P$. Put $\alpha = \beta \eta$. Then this $\alpha$ possesses the required property: $\alpha^{-1} \mathfrak{I}_1 \alpha = \eta^{-1} \beta^{-1} \mathfrak{I}_1 \beta \eta = M(0), \alpha^{-1} \mathfrak{I}_2 \alpha = \gamma^{-1} M(0) \gamma = M(c_k - c_i)$.

**Lemma 3.** There exist two regular elements $\alpha, \beta$ in $A$ such that

$$\alpha \beta \mathfrak{A} = M(0), \quad \alpha \beta \mathfrak{B} = M(d_k - c_i); \quad c_1 \geq c_2 \geq \cdots \geq c_r, \quad d_1 \geq d_2 \geq \cdots \geq d_r.$$

**Proof.** Let $\mathfrak{I}_1, \mathfrak{I}_2 \{\mathfrak{I}_3, \mathfrak{I}_4\}$ be the left and the right orders of $[b]$. According to the above lemma there exist $\gamma, \beta$ such that $\gamma^{-1} \mathfrak{I}_1 \gamma = \beta^{-1} \mathfrak{I}_2 \beta = M(0), \gamma^{-1} \mathfrak{I}_3 \gamma = M(c_k - c_i), \beta^{-1} \mathfrak{I}_4 \beta = M(d_k - d_i)$. $\gamma^{-1} \alpha \beta$ is a two-sided ideal of $M(0)$ and has a form $P^a M(0)$. Put $\alpha = (\gamma P^a)^{-1}$. Then $\alpha \beta \mathfrak{A} = M(0)$. Moreover, $\alpha \beta \mathfrak{B}$ is of a form $M(d_k - c_i + b)$ (Lemma 1). We put $d_k = d_k + b$, and this completes the proof.

We note further that the left order of an ideal $M(a_{ik})$ is $M(b_{ik})$ where $b_{ik} = \max_i (a_{ij} - a_{kj})$.

After these preliminaries our theorems are easy to prove. In Theorem 1 we may, according to Lemma 2, assume that $\mathfrak{I}_1 = M(0), \mathfrak{I}_2 = M(c_k - c_i)$, $c_1 \geq \cdots \geq c_r$. Suppose $\mathfrak{I}_1 \cap \mathfrak{I}_2 = \mathfrak{I}_3$ and $\mathfrak{I}_4$. Since

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5 Chevalley, Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität, vol. 10 (1934), p. 87.
that \( \mathfrak{I}_3 \cap \mathfrak{I}_2 \leq \mathfrak{I}_3, \mathfrak{I}_4 \), it follows that \( \mathfrak{I}_3, \mathfrak{I}_4 \) have the form \( \mathfrak{I}_3 = M(d_k - d_i), \mathfrak{I}_4 = M(f_k - f_i) \). Moreover \( \max (d_k - d_i, f_k - f_i) = \max (0, c_k - c_i) \), \( (i, k = 1, 2, \ldots, r) \). This implies \( \max (d_k - d_i, f_k - f_i) = 0 \) if \( i \geq k \), whence \( d_1 \geq \cdots \geq d_r \) and \( f_1 \geq \cdots \geq f_r \). On applying the same relation to \( i = 1, k = r \), we find that either \( d_1 = d_r \) or \( f_1 = f_r \). In the first case we have \( d_1 = \cdots = d_r, f_1 - f_i = c_1 - c_i, (i = 1, 2, \ldots, r) \), whence \( \mathfrak{I}_1 = \mathfrak{I}_3, \mathfrak{I}_2 = \mathfrak{I}_4 \). The second case gives of course \( \mathfrak{I}_1 = \mathfrak{I}_4, \mathfrak{I}_2 = \mathfrak{I}_3 \).

The assertion about the sum follows now from Theorem 2, which is in turn contained in Theorems 3 and 4.

As to Theorem 3 we notice first that if \( \alpha, \beta \) are two regular elements, the ideals \( \alpha \varepsilon \alpha^{-1}, \beta^{-1} \delta \beta, \beta^{-1} t' \beta \) have the same significance for \( \alpha \beta \delta \) and \( \alpha \beta \delta \) as the ideals \( \tau, \delta, t, t' \) have for \( \alpha \) and \( \beta \). Hence it is sufficient, by Lemma 3, to consider the case where

\[
a = M(0), \quad b = M(d_k - c_i); \quad c_1 \geq c_2 \geq \cdots \geq c_r, \quad d_1 \geq d_2 \geq \cdots \geq d_r.
\]

Then \( \alpha \cap b = M(\max (0, d_k - c_i)) \) and \( \nu = M(a_{ik}) \) with

\[
a_{ik} = \max (0, d_i - c_i) - \max (0, d_i - c_k)
\]

where \( f_i = -\max (0, d_i - c_i), g_i = -\max (0, d_i - c_i) \). Since \( f_k - f_i \geq 0 \) or \( \leq g_k - g_i \) according as \( i \geq k \) or \( i \leq k \), we find that \( \nu \) is the intersection of the two maximal orders \( M(f_k - f_i) \) and \( M(g_k - g_i) \). Further, if we put \( \gamma = \sum \epsilon_{ik} \nu^{-\epsilon_{ik}}, \delta = \sum \epsilon_{ik} \nu^{-\delta_{ik}}, \) then \( \tau = \gamma P^a M(0) \) and \( \delta = M(0)P^{a - \delta} \), whence \( \tau P^{a - a} M(0), \delta = P^{a - a} M(0) \). From this we can easily verify the precise characterization of \( \nu \) given in the theorem.

The part on the sum \( (a, b) \) can be shown by a similar computation. And indeed from that computation we obtain the last assertion in the theorem.

Finally, to prove Theorem 4 we observe again that we have only to consider where \( a, b \) have the form (1). \( a \cap b = M(\max (0, d_k - c_i)), (a^{-1}, b^{-1}) = M(\min (0, c_k - d_i)) \) because \( b^{-1} = M(c_k - d_i) \), and here we notice that \( \max (0, d_k - c_i) = -\min (0, c_i - d_k) \). The third normal ideal \( c \) can be expressed as \( c = \tau^{-1} M(0)\sigma \) with regular elements \( \sigma = \sum \epsilon_{ik} \delta_{ik}, \tau = \sum \epsilon_{ik} k_{ik} \). Let \( P^{a_{ik}} \) be the exact power of \( P \) which divides \( s_{ik}, P^{c_{ik}} \parallel s_{ik}; \) if \( s_{ik} = 0 \) we put \( c_{ik} = \infty \). Let similarly \( P^{d_{ik}} \parallel i_{ik} \). It is evident that \( M(a_{ik}) \), with a system of rational integers \( a_{ik} \), contains \( c^{-1} = M(0) \tau \) if and only if

\[
c_{ij} + d_{ik} \geq a_{ik}, \quad \text{for all } i, j, k, l.
\]
Hence, if we show that the same condition is also necessary and sufficient in order that \( M(-a_{ki}) \subseteq c \), then we will be through. But this is also easy to see. For, \( c = \tau^{-1}M(0)\sigma^{-1} \) consists of all \( \eta = \sum \epsilon_{ik}y_{ik} \). On taking a pair \((j, l)\) of indices, let us consider those \( \eta \) such that \( y_{ik} = 0 \) for \((i, k) \neq (j, l)\). In other words, we consider the equation \( \tau^{-1}(\sum \epsilon_{ik}x_{ik})\sigma^{-1} = \epsilon_{jl}y_{jl} \). But this is equivalent to \( \sum \epsilon_{ik}x_{ik} = \tau\epsilon_{jl}y_{jl}\sigma \), or

\[
x_{ik} = t_{ij}y_{jl}S_{ik}, \quad i, k = 1, 2, \ldots, r.
\]

Suppose now \( M(-a_{ki}) \subseteq c \). Then (3) with \( y_{jl} = P^{-a_{ji}} \) must have a solution \( x_{ik} \in I \). Hence \( 0 \leq d_{ij} - a_{ij} + c_{ik} \) (for all \( i, k \)). Since \((j, l)\) was an arbitrary pair of indices, we have thus established (2). Assume conversely (2). Then obviously \( x_{ik} = t_{ij}P^{-a_{ji}}S_{ik} \in I \) whence \( \epsilon_{jl}P^{-a_{ji}} \in c \) and \( M(-a_{ki}) \subseteq c \).

A second proof of the last part of Theorem 3 is as follows: We observe first that every ideal \( m \) in \( \mathcal{A} \) is additively generated by regular elements contained in \( m \). For, if \( \xi \in m \) we take a scalar element \( a \) (\( \not\in F \)) in \( m \) different from all the characteristic roots of the matrix which represents \( \xi \) in a faithful representation of \( \mathcal{A} \). Then \( \xi - a \) is evidently a regular element and \( \xi = (\xi - a) + a \). Now, let \( \alpha \) be any regular element from the left order of \( a \cap b ; \alpha(a \cap b) \subseteq a \cap b \). Since \( \alpha a \) and \( \alpha b \) are normal ideals, we have, from Theorem 4, \( (a^{-1} \alpha^{-1}, b^{-1} \alpha^{-1}) \supseteq a^{-1}, b^{-1} \) whence \( (a^{-1}, b^{-1}) \alpha^{-1} \supseteq (a^{-1}, b^{-1}) \alpha \). This shows that the left order of \( a \cap b \) is contained in the right order of \( (a^{-1}, b^{-1}) \). But the converse can be seen in quite a similar manner.

Remark. The structure of the residue class algebra \( 3_1 \cap 3_2/\rho(3_1 \cap 3_2) \) is easy to analyze, but perhaps does not deserve a detailed discussion. We merely note that the algebra is not symmetric, in fact is not weakly symmetric, except for the trivial case \( (3_1)_p = (3_2)_p \); this remark may be of some interest in view of a recent paper by R. Brauer.

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