ON THE SUPPORTING-PLANE PROPERTY OF A
CONVEX BODY

DAVID MOSKOVITZ AND L. L. DINES

In an earlier paper, the authors have shown that in a linear space \( \mathcal{S} \) with an inner product, a set \( \mathcal{M} \) which is closed and linearly connected is supported at a set of boundary points which is everywhere dense on the boundary of \( \mathcal{M} \), and an example is given to show that such a set \( \mathcal{M} \) may have boundary points through which no supporting plane exists. The purpose of this paper is to show that if a set, in addition to being linearly connected and closed, also possesses inner points, then it is completely supported at its boundary points. In (I), reference was made to a paper by Ascoli in which such a result was obtained in a separable space. We do not assume our space \( \mathcal{S} \) to be separable. The definitions and results of (I) will be used in this paper.

A set \( \mathfrak{R} \), which is a proper subset of the space \( \mathcal{S} \), will be called a convex body if it is linearly connected, closed, and possesses inner points. In the sequel \( \mathfrak{R} \) will always denote a convex body.

With reference to the set \( \mathfrak{R} \), there is associated with each point \( x \) of the space \( \mathcal{S} \) a nonnegative number \( r(x) \): if \( x \) is an inner point of \( \mathfrak{R} \), \( r(x) \) is defined as the least upper bound of the radii of spheres about \( x \) which do not contain points exterior to \( \mathfrak{R} \); for other points of \( \mathcal{S} \), \( r(x) \) is defined to be zero. We will call \( r(x) \) the radius at the point \( x \).

If \( x_1 \) is a point of \( \mathfrak{R} \), all points \( x \) of the sphere \( \| x-x_1 \| \leq r(x_1) \) are points of \( \mathfrak{R} \).

**Theorem 1.** Let \( r_1 \) and \( r_2 \) be the radii at the points \( x_1 \) and \( x_2 \), respectively, of the convex body \( \mathfrak{R} \). Then the radius \( r \) at the point

\[
x = x_1 + k(x_2 - x_1),
\]

satisfies

\[
r \geq r_1 + k(r_2 - r_1).
\]

**Proof.** Let \( y = x + \rho u \), where \( \rho = r_1 + k(r_2 - r_1) \) and \( \| u \| = 1 \). The points \( y_1 = x_1 + r_1 u \) and \( y_2 = x_2 + r_2 u \) are points of \( \mathfrak{R} \). But from the definitions of \( x \), \( \rho \), and \( y \), it follows that \( y = y_1 + k(y_2 - y_1) \). Hence \( y \), being on the segment joining \( y_1 \) and \( y_2 \), is also a point of \( \mathfrak{R} \). Consequently

---

1 Presented to the Society, September 5, 1939.

all points on the boundary of the sphere with radius \( \rho \) and center \( x \) are in \( \mathcal{S} \). Since \( \mathcal{S} \) is linearly connected, also all points within this sphere are in \( \mathcal{S} \). Therefore \( r \geq \rho \) and the theorem is established.

The following corollaries, which appear self-evident in ordinary space, can be shown to be direct consequences of the preceding theorem.

**Corollary 1.** Each point of the segment joining two inner points of \( \mathcal{S} \) is an inner point of \( \mathcal{S} \).

**Corollary 2.** If \( x_0 \) is a boundary point and \( x_1 \) an inner point of \( \mathcal{S} \), then the points \( x = x_1 + k(x_0 - x_1) \) are inner points of \( \mathcal{S} \) for \( 0 \leq k < 1 \), and exterior points for \( k > 1 \).

With reference to a given boundary point \( x_0 \) of the set \( \mathcal{S} \), there is associated with each point \( x \), other than \( x_0 \), of the space \( \mathbb{R} \) a non-negative number \( s(x) \), defined by

\[
s(x) = \frac{r(x)}{\|x - x_0\|}.
\]

If \( x \) is an exterior point or a boundary point of \( \mathcal{S} \), other than \( x_0 \), \( s(x) \) is equal to zero; if \( x \) is an inner point of \( \mathcal{S} \), \( s(x) \) is positive; \( s(x_0) \) is not defined.

It is also obvious that \( s(x) \leq 1 \), since \( r(x) \leq \|x - x_0\| \).

**Theorem 2.** Let \( x_0 \) be a given boundary point of the convex body \( \mathcal{S} \), and let \( x_t \) be given by

\[
x_t = x_0 + tu,
\]

where \( t > 0 \), \( \|u\| = 1 \).

Then, for fixed \( u \),

(a) \( s(x_t) \) is a non-decreasing function as \( t \to 0 \); and

(b) \( \lim_{t \to 0} s(x_t) \) exists.

**Proof.** In case there are no points of \( \mathcal{S} \) given by (1), the theorem is obviously true, for then

\[
s(x_t) = 0 \quad \text{for } t > 0, \quad \lim_{t \to 0} s(x_t) = 0.
\]

In case there are points of \( \mathcal{S} \) given by (1), let \( x_1 \) and \( x_2 \) be two points of \( \mathcal{S} \) on (1) for parameter values \( t_1 \) and \( t_2 \), where \( t_1 < t_2 \); then we have

\[
\begin{align*}
x_1 &= x_0 + t_1 u, \quad x_2 = x_0 + t_2 u; \\
s(x_1) &= \frac{r(x_1)}{t_1}, \quad s(x_2) = \frac{r(x_2)}{t_2}.
\end{align*}
\]

\(^3\) Since \( s \) is a function of \( x_0 \) as well as \( x \), a more explicit notation would be \( s(x_0, x) \); but the simpler notation will suffice, inasmuch as the function is to be used in the sequel only with reference to a fixed boundary point \( x_0 \).
But \( x_1 = x_0 + (t_1/l_2)(x_2 - x_0) \), and hence, by Theorem 1, we have
\[
r(x_1) \geq \frac{t_1}{l_2} r(x_2),
\]
since \( r(x_0) = 0 \). Therefore, by (2), \( s(x_1) \geq s(x_2) \).

This result establishes part (a) of the theorem. Since \( s(x_t) \) cannot exceed one, obviously part (b) of the theorem is true.

Let \( \Sigma \) be the unit sphere about \( x_0 \); and let \( p_u \) be the point on \( \Sigma \) given by \( p_u = x_0 + u, \| u \| = 1 \). Let \( x_t = x_0 + tu, \ (0 < t < 1) \), be the segment joining \( x_0 \) to \( p_u \); and let
\[
\sigma(u) = \lim_{t \to 0} s(x_t).
\]

We thus have a function \( \sigma(u) \) uniquely defined at each point \( p_u \) on the sphere \( \Sigma \). Obviously, by its definition, we have
\[
0 \leq \sigma(u) \leq 1.
\]
Also \( \sigma(u) = 0 \) only if the segment joining \( x_0 \) to \( p_u \) does not contain any inner points of \( \mathcal{K} \). If the segment joining \( x_0 \) to \( p_u \) contains inner points of \( \mathcal{K} \), we have \( \sigma(u) > 0 \).

**Lemma 1.** Let \( p_u \) and \( p_v \) be two points on \( \Sigma \), such that
\[
p_u = x_0 + u, \quad p_v = x_0 + v, \quad v = -u.
\]
Then at least one of the numbers \( \sigma(u) \) or \( \sigma(v) \) is equal to zero.

**Proof.** Assume \( \sigma(u) > 0 \); then the segment joining \( x_0 \) to \( p_u \) contains inner points. Consequently, by Corollary 2, the segment joining \( x_0 \) to \( p_v \) does not contain any inner points. Therefore, \( \sigma(v) = 0 \).

**Theorem 3.** Let \( x_0 \) be a given boundary point of the convex body \( \mathcal{K} \), and let \( \Sigma \) be the unit sphere about \( x_0 \). Let \( p_u \) and \( p_v \) given by
\[
p_u = x_0 + u, \quad \| u \| = 1, \quad p_v = x_0 + v, \quad \| v \| = 1
\]
be two distinct points on \( \Sigma \), for which \( \sigma(u) \) and \( \sigma(v) \) are both positive. Then there exists a point \( p_w \) distinct from \( p_u \) and \( p_v \) for which
\[
\sigma(w) > \frac{1}{2} [\sigma(u) + \sigma(v)].
\]

*The limit was shown to exist in Theorem 2; we are denoting the value of this limit by \( \sigma(u) \). It may be of interest to note that
\[
\sigma(u) = \lim_{x \to x_0, \ x_t = x_0 + tu} s(x) = \lim_{x \to x_0} \frac{r(x) - r(x_0)}{\| x - x_0 \|} = r'_u(x_0)
\]
is the directional derivative of \( r(x) \) at \( x_0 \) in the direction \( u \).
PROOF. Let \( x_t \) and \( y_t \) be points of \( \mathbb{R} \) given by
\[
x_t = x_0 + tu, \quad y_t = x_0 + tv, \quad 0 < t < 1,
\]
and let \(((u, v)) = \lambda\). Then, certainly \(|\lambda| \leq 1\). But if \( \lambda = 1 \), \( u = v \), and \( p_u \) and \( p_v \) are not distinct. If \( \lambda = -1 \), \( u = -v \), in which case not both of the numbers \( \sigma(u) \) and \( \sigma(v) \) can be positive, because of Lemma 1. Consequently, we have
\[
-1 < \lambda < 1.
\]
Let \( z_t = \frac{1}{2}(x_t + y_t) \); then \( z_t = x_0 + \xi tw \), where \( ||w|| = 1 \) and \( \xi = \frac{1}{2}(1 + \lambda)^{1/2} \). Thus
\[
0 < \xi < 1.
\]
We thus have a point \( p_w \) on the sphere \( \Sigma \) defined by \( p_w = x_0 + w \). Now,
\[
r(z_t) \geq \frac{1}{2} [r(x_t) + r(y_t)], \quad \text{by Theorem 1.}
\]
Hence
\[
s(z_t) = \frac{r(z_t)}{\xi t} \geq \frac{1}{2\xi} \left[ \frac{r(x_t)}{t} + \frac{r(y_t)}{t} \right] = \frac{1}{2\xi} [s(x_t) + s(y_t)],
\]
and
\[
\lim_{t \to 0} s(z_t) \geq \frac{1}{2\xi} \lim_{t \to 0} [s(x_t) + s(y_t)],
\]
from which
\[
\sigma(w) \geq \frac{1}{2\xi} [\sigma(u) + \sigma(v)] > \frac{1}{2} [\sigma(u) + \sigma(v)].
\]
Thus the theorem is established.

Let \( \tilde{\sigma} \) denote the least upper bound of the function \( \sigma(u) \) as \( p_u \) varies over the sphere \( \Sigma \). Then, also \( 0 \leq \tilde{\sigma} \leq 1 \); and \( \tilde{\sigma} = 0 \) is possible only for sets which do not have any inner points. For a convex body \( \mathbb{R} \), we have \( 0 < \tilde{\sigma} \leq 1 \).

In the material which follows, it is to be understood that \( x_0 \) is a fixed boundary point of the convex body \( \mathbb{R} \), \( s(x) \) is defined relative to \( x_0 \), \( \Sigma \) is the unit sphere about \( x_0 \), \( \sigma(u) \) is the function defined above on the boundary of \( \Sigma \), and \( \tilde{\sigma} \) the least upper bound of \( \sigma(u) \) on \( \Sigma \).

**Theorem 4.** If there is a point \( p_u \) on \( \Sigma \) for which \( \sigma(u) = \tilde{\sigma} \), this point is unique.

**Proof.** Suppose, if possible, that there were a second point \( p_v \) for which \( \sigma(v) = \tilde{\sigma} \). Then, by Theorem 3, since \( \tilde{\sigma} > 0 \), there would be a point \( p_w \) for which
\[ \sigma(w) > \frac{1}{2} [\sigma(u) + \sigma(v)] = \bar{\sigma}. \]

But since no \( \sigma(w) \) can exceed \( \bar{\sigma} \), there cannot be a second point \( p_\ast \) of the type described.

**THEOREM 5.** Let \( p_u \) be a point on \( \Sigma \) for which \( \sigma(u) = \bar{\sigma} \). If \( v \) satisfies the conditions \( ||v|| = 1 \) and \( ((u, v)) < 0 \), then the points \( z_t = x_0 + tv, t > 0 \), are exterior points of \( \mathcal{F} \).

**PROOF.** Let \( p_u = x_0 + u, ||u|| = 1 \), and \( p_v = x_0 + v, ||v|| = 1 \); and let \( ((u, v)) = -\lambda \), where \( \lambda > 0 \). Assume, if possible, that there is a point \( z = x_0 + dv, d > 0 \), belonging to \( \mathcal{F} \). Let \( w \) be the projection (defined in (I)) of \( z \) on the line through \( x_0 \) and \( p_u \). Then

\[ w = p_u + c(x_0 - p_u), \]

where

\[ c = \frac{((z - p_u, x_0 - p_u))}{||p_u - x_0||^2} = ((z - x_0 - u, -u)) \]

\[ = ((dv - u, -u)) = 1 + \lambda d. \]

Hence,

\[ w = p_u - (1 + \lambda d)u = x_0 - \lambda du. \]

On the segment joining \( x_0 \) to \( p_u \), let \( x_t = x_0 + tu \) be an inner point of \( \mathcal{F} \). Let \( y_t \) be the projection of \( x_0 \) on the line through \( x_t \) and \( z \). Then

\[ y_t = x_t + k(z - x_t) \]

where

\[ k = \frac{((x_0 - x_t, z - x_t))}{||z - x_t||^2} = \frac{((-tu, dv - tu))}{||z - x_t||^2} = \frac{\lambda(td + t^2)}{d^2 + 2\lambda td + t^2}, \]

since \( z - x_t = z - x_0 + x_0 - x_t = dv - tu \) and

\[ ||z - x_t||^2 = d^2 - 2t((u, v)) + t^2 = d^2 + 2\lambda td + t^2. \]

From the above value of \( k \), it is easily seen that \( 0 < k < 1 \), which means that \( y_t \) is a point of \( \mathcal{F} \). The following are easily established:

\[ ||y_t - x_0||^2 = \frac{t^2d^2(1 - \lambda^2)}{d^2 + 2\lambda td + t^2} \neq 0, \]

since \( \lambda \neq \pm 1 \), and

\[ ||z - w||^2 = d^2(1 - \lambda^2). \]
From these, and previous results, we obtain
\[
\frac{\|x_t - x_0\|^2}{\|y_t - x_0\|^2} = \frac{t^2(d^2 + 2\lambda dt + t^2)}{t^2d^2(1 - \lambda^2)} = \frac{d^2 + 2\lambda dt + t^2}{d^2(1 - \lambda^2)} = \frac{\|z - x_t\|^2}{\|z - w\|^2}.
\]
Therefore, we have
\[
\frac{\|x_t - x_0\|}{\|y_t - x_0\|} = \frac{\|z - x_0\|}{\|z - w\|}. \tag{3}
\]
Now \(s(y_t) = r(y_t)/\|y_t - x_0\|\) and \(s(x_t) = r(x_t)/\|x_t - x_0\|\), where \(r(y_t)\) and \(r(x_t)\) denote the radii at the points \(y_t\) and \(x_t\), respectively. Also \(r(y_t) \geq (1 - k)r(x_t)\), by Theorem 1 and the definition of \(y_t\). Hence
\[
s(y_t) = s(x_t) \geq \frac{r(y_t)}{r(x_t)} \cdot \frac{\|x_t - x_0\|}{\|y_t - x_0\|} = (1 - k) \frac{\|x_t - x_0\|}{\|y_t - x_0\|}
\]
\[
= (1 - k) \frac{\|z - x_0\|}{\|z - w\|}, \tag{4}
\]
the last equality being a consequence of (3).

But \(k = \|y_t - x_t\|/\|z - x_t\|\) and \(1 - k = \|z - y_t\|/\|z - x_t\|\). Therefore, from (4), we have
\[
s(y_t) \geq \frac{\|z - y_t\|}{\|z - w\|} s(x_t). \tag{5}
\]
Now,
\[
\lim_{t \to 0} \frac{\|z - y_t\|}{\|z - w\|} = \frac{d}{d(1 - \lambda^2)^{1/2}} = \frac{1}{(1 - \lambda^2)^{1/2}} > 1,
\]
since \(z\) and \(w\) are independent of \(t\), while \(y_t \to x_0\) as \(t \to 0\). Therefore, from (5),
\[
\lim_{t \to 0} s(y_t) \geq \frac{1}{(1 - \lambda^2)^{1/2}} \sigma(u) > \sigma(u) = \tilde{\sigma}.
\]
But this is impossible; hence the assumption that \(z\) was a point of \(\mathcal{K}\) is untenable.

**Theorem 6.** Let \(p_u\) be a point on \(\Sigma\) for which \(\sigma(u) = \tilde{\sigma}\). Then the plane
\[
\pi(x) \equiv \langle (u, x - x_0) \rangle = 0 \tag{6}
\]
is a supporting plane of \(\mathcal{K}\) through the boundary point \(x_0\).

**Proof.** If the plane (6) were not a supporting plane, there would be
a point $z$ of $\mathcal{K}$ for which $\pi(z) < 0$. Let $v = (z - x_0)/\|z - x_0\|$; then

$$((u, v)) = \frac{\pi(z)}{\|z - x_0\|} < 0, \quad z = x_0 + \|z - x_0\|v.$$ 

But, $v$ satisfies the conditions of Theorem 5; therefore $z$ must be an exterior point of $\mathcal{K}$. Consequently, there cannot be a point $z$ of $\mathcal{K}$ for which $\pi(z) < 0$; and (6) is indeed a supporting plane.

**Theorem 7.** Let $x_0$ be a given boundary point of the convex body $\mathcal{K}$, and let $\Sigma$ be the unit sphere about $x_0$. There is a unique point $\rho_\alpha$ on $\Sigma$ for which $\sigma(\rho_\alpha) = \alpha$.

**Proof.** We have only to show the existence of one point $\rho_\alpha$ for which $\sigma(\rho_\alpha) = \alpha$. The uniqueness of this point will follow from Theorem 4. 

From the definition of $\sigma$ it follows that for any preassigned $\epsilon > 0$, there exists a point on $\Sigma$ for which the value of $\sigma$ is greater than $\sigma - \epsilon$. Choose a monotone decreasing sequence of positive numbers $\{\epsilon_n\}$ with limit zero. Corresponding to each $\epsilon_n$ there exists a point $\rho_{\alpha_n}$ on $\Sigma$ for which $\sigma(\rho_{\alpha_n}) > \sigma - \epsilon_n$. We wish to show that the sequence of points $\{\rho_{\alpha_n}\}$ on $\Sigma$ converges.

Let $\rho_{\alpha_n} = x_0 + u_n$, $\|u_n\| = 1$, and $\rho_{\alpha_m} = x_0 + u_m$, $\|u_m\| = 1$. Then

$$\|\rho_{\alpha_n} - \rho_{\alpha_m}\|^2 = 2 - 2((u_n, u_m)).$$

Let $w = \frac{\alpha}{\xi} (u_n + u_m)$, where $\xi$ is so chosen that $\|w\| = 1$. Then we have

$$\xi^2 = \frac{1}{2} \left[ 1 + ((u_n, u_m)) \right].$$

Let $\rho_w = x_0 + w$; from the proof of Theorem 3, we know that

$$\sigma(w) \geq \frac{1}{2\xi} \left[ \sigma(u_n) + \sigma(u_m) \right] \geq \frac{1}{2\xi} \left[ 2\sigma - \epsilon_n - \epsilon_m \right].$$

But $\sigma \geq \sigma(w)$; hence $\sigma > (1/\xi) \left[ \sigma - (\epsilon_n + \epsilon_m)/2 \right]$, from which

$$\xi^2 > \left[ 1 - \frac{1}{2\sigma} (\epsilon_n + \epsilon_m) \right]^2.$$

Using the value of $\xi^2$ from (8) we easily find that

$$((u_n, u_m)) > 2 \left[ 1 - \frac{1}{2\sigma} (\epsilon_n + \epsilon_m) \right]^2 - 1.$$

Then using (7), we obtain

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
\[(9) \quad \|p_{u_n} - p_{u_m}\|^2 < \frac{4}{\bar{\sigma}} (\epsilon_n + \epsilon_m) - \frac{1}{\bar{\sigma}^2} (\epsilon_n + \epsilon_m)^2.\]

Since \(\lim_{n,m \to \infty} ||p_{u_n} - p_{u_m}|| = 0\) and the space \(\mathcal{S}\) is complete, as was assumed in (I) and throughout this paper, the sequence \(\{p_{u_n}\}\) converges to a point \(p_u\). This point \(p_u\) is on \(\Sigma\), and moreover \(\sigma(u) = \bar{\sigma}\), since it is easily shown that \(\sigma(u)\) is greater than \(\bar{\sigma} - \epsilon\) for any pre-assigned positive \(\epsilon\).

**Theorem 8.** A convex body \(K\) is completely supported at its boundary points.

**Proof.** Let \(x_0\) be a boundary point of \(K\). There exists a point \(p_u\) on the unit sphere \(\Sigma\) about \(x_0\) for which \(\sigma(u) = \bar{\sigma}\), by Theorem 7. Hence the plane \((u, x - x_0) = 0\) is a supporting plane of \(K\) through \(x_0\), by Theorem 6. Since similar statements can be made for each boundary point, \(K\) is completely supported at its boundary points.

From the material above, the following additional result may be established without much difficulty:

**Corollary 3.** There exists a unique supporting plane through the boundary point \(x_0\) of the convex body \(K\) only if \(\bar{\sigma} = 1\); for \(\bar{\sigma} < 1\), there is an infinite number of supporting planes through \(x_0\).

A primary classification of boundary points of a convex body may thus be made in terms of \(\bar{\sigma}\), which is a function defined over the boundary of the convex body.

**Carnegie Institute of Technology**