A THEOREM ON THE ROTATION GROUP OF THE TWO-SPHERE[1]

DEANE MONTGOMERY AND LEO ZIPPIN

Let R be the group of all rotations of euclidean three-dimensional space E about its origin. We shall take the domain of operation of R to be a euclidean two-sphere S with center at the origin. On the space S, R is transitive. The group R is topological and, considered as a space, is homeomorphic to projective three-space.

While studying the action of groups in certain spaces, the following theorem, which, as far as we know, is not in the literature, occurred to us.

THEOREM. *Let G be any proper subgroup of R. Then G is not transitive on S.*

The subgroup G is subject to no restrictions whatever; in particular it is not closed. The proof will be essentially topological and we begin by assuming that G is transitive on S.

The identity element e of G must be a limit point of G. For otherwise G would be finite and hence certainly not transitive on S. Let g be an element of G which is near e. The element g has a pair of fixed points on S one of which will be denoted by p. The element g is a rotation through a small angle A around the line through p and the origin. Under this rotation points near p move in a certain direction around p, say the clockwise direction.

Now let x be any point of S, and let h be any element of G such that $h(p)=x$. The element hgh^{-1} is in G, and it is a rotation through the angle A around the line through x and the origin. Points near x will be moved in a clockwise direction around x.

We have therefore shown that for every point x of the sphere, G contains a rotation through the angle A around the line through the origin and x. For each x, points near x are moved by the associated rotation in a specified sense around x. Let M be the totality of all these rotations the existence of which has just been demonstrated. The set M is homeomorphic to S and consequently is a two-sphere; furthermore M is a subset of G.

Consider the elements of G given by $g^{-1}M$. This set is a two-sphere passing through the identity element. There will be an arc L, which is in $g^{-1}M$ and therefore in G, leading from e to some element of G,

[1] Presented to the Society, October 28, 1939, under the title *Note on rotation-group of the two-sphere.*

call it g' distinct from e. Let U be an open three-cell subset of R which includes e. Let N be a two-sphere made up of elements of G. The existence of arbitrarily small two-spheres of this kind is proved as above by choosing the element g sufficiently near to e, and we may assume that N is in U. We may also assume that LN is in U and that $g'N$ is outside N: as must be the case if N is small enough.

The arc L may now be used to define a deformation of N to $g'N$. Under this deformation all points swept out by N are in G. Furthermore every point inside N is swept out by the deformation. Hence every point of R inside N is in the group G. The group G is thus seen to contain open subsets and, because of homogeneity, G is open in R. It must therefore coincide with R. The assumption that a proper subgroup G was transitive on S has now led to a contradiction, and the proof is therefore complete.

SMITH COLLEGE AND
QUEENS COLLEGE

ON ORDERED ALGEBRAS[1]

A. A. ALBERT

In his first Madison Colloquium lecture M. H. Stone connected the theory of convex bodies with linear sets over an ordered field. It was natural then to ask whether his theory could be extended by replacing these fields by ordered rings and indeed to ask whether there exist ordered rings which are not fields. I discussed this question at that time with S. MacLane and we attempted to answer it. MacLane has since found an example,[2] in the literature, of a noncommutative ordered quasi-field. It is not an algebra (of finite order) however and it is my purpose in this note to give a very brief proof in elementary language of the following decisive result.

THEOREM. *Every ordered algebra is a field.*

We first observe some known consequences of the order postulates.[3] The postulates on products imply that an ordered ring contains no divisors of zero and hence that every ordered algebra is a division algebra D. Then D has a unity quantity $1=1^2>0$, the sums

[1] Presented to the Society, December 2, 1939.

[2] Cf. Reidemeister, *Grundlagen der Geometrie*, p. 40. It is also shown in this text that archimedian ordered quasi-fields are fields.

[3] The order postulates on page 40 of my *Modern Higher Algebra* were called postulates for an ordered field but are valid for arbitrary rings.