$$(18) \qquad R'_{n-p_1} = 0, \qquad R'_{n-p_2} = 0, \qquad \cdots, \qquad R'_{n-p_k} = 0$$

for an arbitrary original polygon P. Further, no other relations $R'_{n-p}=0$ ($p \neq p_1$ or $p_2 \cdots$ or p_k) are satisfied by P' if P remains general (P' has no higher than the kth degree of regularity). This is also seen from (16′), where $\phi(\omega^p) \neq 0$, $R_{n-p} \neq 0$ (since P is general); therefore $R'_{n-p} \neq 0$.

In fact, no relations of any kind besides (18) are satisfied by $P' = MP$ if P remains general. This is because, by the general theory of systems of linear equations, it can be readily shown that if the conditions (17) are satisfied by the coefficients α in (2), then the conditions (18) are sufficient as well as necessary in order that (2) be solvable for the z's in terms of the z''s. This is to say that for *any* polygon P' obeying (18) a polygon P can be found such that $P' = MP$; indeed, the class of such polygons P depends linearly on k complex parameters.

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AXIOMS FOR MOORE SPACES AND METRIC SPACES[1]

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We shall consider a set of five axioms in terms of the undefined notions of *point* and *region*. It will be shown that these axioms are independent and that they constitute a set of conditions necessary and sufficient for a space to be a complete metric space. It will also be shown that certain subsets of this set of axioms constitute necessary and sufficient conditions for a space to be (1) a metric space, (2) a Moore space, (3) a complete Moore space. Axiom 2 and a more general form of Axiom 1 have been stated by the author in an earlier paper [**1**]. Following terminology of F. B. Jones [**2**], a space is said to be a *Moore space* provided conditions (1), (2), and (3) of Axiom 1 (that is, Axiom 1_0) of R. L. Moore's *Foundations of Point Set Theory* [**3**] are satisfied. A space is said to be a *complete Moore space* provided it satisfies all the conditions of that axiom. Wherever the notion of region is employed, whether as a defined or an undefined notion, it is understood that a necessary and sufficient condition that a point P be a limit point of a point set M is that every region containing P contain a point of M distinct from P. The letter S is used to denote the set of all points.

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AXIOM 0. *Every region is a point set.*

AXIOM 1. *There exists a countable family F such that* (1) *every element of F is a collection of regions covering S,* (2) *if R is a region and A and B are points of R, there exists a collection G of F such that if g is a region of G that contains A, $\bar{g}$ is a subset of $(R-B)+A$.*

AXIOM 2. *If P is a point and H and K are regions containing P, there exists a region R containing P which is a subset both of the region H and of K.*

AXIOM 3. *If α is a monotonic descending sequence of closed point sets $A_1, A_2, \cdots$ such that for each n there exists a monotonic descending sequence ρ_n of distinct regions $R_1, R_2, \cdots, R_n$ containing A_n, then there exists a point common to all the elements of α.*

AXIOM 4. *If G is a collection of regions covering S, there exists a collection H of regions covering S such that if h_1 and h_2 are intersecting regions of H, then (h_1+h_2) is a subset of a region of G.*

THEOREM 1. *In order that a space be a Moore space, it is necessary and sufficient that it satisfy Axioms* 0, 1, *and* 2.

The necessity of these conditions is evident. We shall undertake to show their sufficiency without changing the notion of region. Let $H_1, H_2, \cdots$ be a type ω sequence of all the elements of family F postulated by Axiom 1. Let G_1 denote the collection of all regions R such that R is a subset of a region of H_1. Let G_2 denote the collection of all regions R such that R is a subset of a region of H_1 and a region of H_2. For each positive integer n let G_n denote the collection of all regions R such that R is a subset of a region of H_i for each $i \leqq n$. For each n, G_n covers S, by Axiom 2. Furthermore, for each n, G_n contains all the regions of G_{n+1}. The sequence $G_1, G_2, \cdots$ satisfies all the conditions of Axiom 1_0 of R. L. Moore.

As a means to proving the next theorem, we shall prove the following lemma on the basis of Moore's Axioms 0 and 1_0:

LEMMA 1. *If M is a set of points and G is a collection of domains covering M, there exists a collection H of domains covering M such that no domain of H is a subset of another domain of H and such that every domain of H is a subset of some domain of G.*

Suppose that M is a set of points and G a collection of domains covering M. For each positive integer n let T_n denote the set of all points P of M such that some domain of G contains every region of G_n that contains P. Then $M=\sum_{n=1}^{\infty} T_n$. For each positive integer n

let θ_n denote a well-ordered sequence of the points of T_n. Let θ denote the sequence obtained by taking first the elements of θ_1, then the elements of θ_2, and so on. Let $t_{i,\mu}$ denote the first element of θ. where i is the smallest integer such that $t_{i,\mu}$ is an element of θ_i and where μ is an ordinal number denoting the order of $t_{i,\mu}$ in θ_i. (Some sets T_n may be vacuous.) We shall now define a sequence Δ of domains $D_1, D_2, \cdots$. Let D_1 denote the sum of all the regions of G_i that contain $t_{i,\mu}$. Let $t_{j,\nu}$ denote the first point of θ not contained in D_1. Let D_2 denote the sum of all the regions of G_j that contain $t_{j,\nu}$. In general, suppose that Δ_α denotes any *abschnitt* of Δ; then let $t_{k,\xi}$ denote the first point of θ not contained in any domain of Δ_α and let D_α denote the sum of all the regions of G_k that contain $t_{k,\xi}$. Let H denote the collection of all the domains of Δ. Then H has the required properties.

THEOREM 2. *In order that a space be a complete Moore space, it is necessary and sufficient that it satisfy Axioms* 0, 1, 2, *and* 3.

We shall first show the sufficiency of these conditions. Let $H_1, H_2, \cdots$ denote a type ω sequence of the elements of family F of Axiom 1. For each positive integer n let G_n denote the collection of all regions R such that R is a point or a proper subset of a region of H_n and of a region of G_{n-1}. It follows, with the help of Axiom 2, that sequence $G_1, G_2, \cdots$ satisfies conditions (1), (2), and (3) of Moore's Axiom 1. It remains to be shown that it satisfies condition (4). Suppose that $M_1, M_2, \cdots$ is a type ω sequence of nondegenerate closed point sets such that for each n, M_n contains M_{n+1} and is a subset of some region of G_n. Let R_n denote a region of G_n that contains M_n. Then R_n is a proper subset of a region R_{n-1} of G_{n-1}. Similarly R_{n-1} is a proper subset of a region R_{n-2} of G_{n-2}. Thus the conditions of Axioms 3 are satisfied and hence there exists a point common to all the sets $M_1, M_2, \cdots$.

We shall now show the necessity of these conditions by redefining region. Let $G_1, G_2, \cdots$ be a sequence of collections of regions postulated by Moore's Axiom 1. For each n let H_n denote a collection of domains covering S such that no domain of H_n is a subset of another domain of H_n and such that every domain of H_n is a subset of a region of G_n. For each n, H_n exists, by Lemma 1. Let F denote the family of all collections H_n. Let $H=\sum_{n=1}^{\omega} H_n$. If the domains of H are called regions and if nothing else is called a region, then Axioms 0, 1, 2, and 3 are satisfied. (1) Clearly Axioms 0 and 1 are satisfied. (2) We shall show that Axiom 2 is satisfied. Let h and k denote two domains of H having a point P in common. There exists an integer n such that every region of G_n that contains P is a subset of $h\cdot k$. Let R denote a domain of H_n. Then R is a subset of some region of G_n and hence of $h\cdot k$. (3) We shall now show that Axiom 3 is satisfied. Let α denote a type ω sequence of closed point sets $A_1, A_2, \cdots$, and for each n let ρ_n denote a type n sequence of domains of H, $R_1, R_2, \cdots, R_n$ satisfying the conditions of Axiom 3. Since for each n no domain of H_n is a subset of another domain of H_n, it follows that there exists an $i\geqq n$ such that some domain of ρ_n belongs to H_i and hence is a subset of a region of G_i. It follows that for each n, A_n is a subset of a region of G_n and hence there exists a point common to all the elements of α.

THEOREM 3. *In order that a metric space be complete it is necessary and sufficient that it satisfy Axiom* 3.

This follows immediately with the aid of Theorem 2 and a result of J. H. Roberts [**4**] to the effect that every metric space that satisfies Axiom 1 of R. L. Moore is complete. In a metric space every interpretation of region that preserves the notion of limit point satisfies Axioms 0, 1, 2, and 4.

THEOREM 4. *In order that a space be metric it is necessary and sufficient that it satisfy Axioms* 0, 1, 2, *and* 4.

We shall first show the sufficiency of these conditions. We have shown that Moore's Axiom 1_0 follows from Axioms 0, 1, and 2. If Axiom 4 be added, it can be shown that the following stronger analogue (due to R. L. Moore) of Moore's Axiom 1_0 follows: "There exists a sequence $G_1, G_2, \cdots$ such that (1) for each n, G_n is a collection of regions covering S, (2) for each n, G_n contains G_{n+1}, (3) if R is a region and A and B are points of R, there exists an integer n such that if h and k are two regions of G_n having a point in common and such that h contains A, then $h+k$ is a subset of $(R-B)+A$." Moore has shown that this proposition is a necessary and sufficient condition for a space to be metric.

We shall show the necessity of these conditions. Suppose that S denotes a space $(\mathcal{D})$. Let all spheroids be called regions. Let collection H_n of family F be the set of all spheroids of radius less than $1/n$. Clearly Axioms 0, 1, and 2 are satisfied. We shall show that Axiom 4 is satisfied. Let G denote a collection of spheroids covering S. For each positive integer n let T_n denote the set of all points P such that there exists a spheroid of G containing the sum of every two-linked chain of spheroids of radius less than $1/n$ that contains P. Then $S=\sum_{n=1}^{\omega}T_n$. Let Q_n denote the collection of all spheroids of radius less than $1/n$ containing a point of T_n. Let $Q=\sum_{n=1}^{\omega} Q_n$. Then Q is the required collection, for the sum of every two-linked chain of regions of Q is a subset of some region of G.

THEOREM 5. *In order that a space be a complete metric space, it is necessary and sufficient that it satisfy Axioms* 0, 1, 2, 3, *and* 4.

This is an immediate consequence of Theorems 3 and 4.

INDEPENDENCE EXAMPLES

For Axiom 1. Let S be the set of all real numbers between 0 and 1. Let p and q denote two real numbers such that $0<p<q<1$. Let the collection of all regions be the collection of all segments ab such that either (1) $0<a<p$ and $q<b<1$, or (2) $0<a<p$ and $0<b<p$, or (3) $q<a<1$ and $q<b<1$.

For Axiom 2. Let S be the set of all points on the x axis between $(-1, 0)$ and $(+1, 0)$. Let every segment of S not containing O (0, 0) or having O as an end point be taken as a region. Furthermore, let every point set consisting of O together with a segment of S having O as an end point be taken as a region. If n is an odd positive integer, let collection H_n of family F of Axiom 1 be the collection of all regions not containing O and of length less than $1/n$, together with all left-hand regions containing O and of length less than $1/n$. If n is even, we have the same statement except that we substitute right-hand regions containing O for left-hand regions.

For Axiom 3. Let S be the set of all rational points on the x axis. Let the sets of all rational points of all segments be called regions.

For Axiom 4. Let S be the set of all points on or above the x axis. Let regions be the interiors of all circles lying wholly above the x axis, together with all point sets Q such that Q is the interior of a circle tangent to the x axis plus the point of tangency. (Example due to R. L. Moore.)

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