ALMOST CYCLIC ELEMENTS AND SIMPLE LINKS OF A CONTINUOUS CURVE[1]

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Many of the definitions and results concerning connected im kleinen continua become useful only when these continua are locally compact. This is especially true in the cyclic element theory. For if a continuous curve M is not locally compact, it is not necessarily true that a simple closed curve in M belongs to a cyclic element of M. Furthermore, if a continuous curve M is not locally compact, it is not necessarily true that a simple closed curve in M belongs to a simple link in M. However, if M is a continuous curve, there are in M certain subcontinua which strongly resemble both cyclic elements and simple links such that if J is a simple closed curve in M, one of them contains J. It is the purpose of this paper to define these sets, to develop a few of their properties, and to show how they are of considerable interest in spaces where the Jordan curve theorem holds true.

1. **Results for complete Moore spaces.** In this section it is assumed that S, the set of all points, is a complete Moore space, that is, Axioms 0 and 1 of R. L. Moore's *Foundations of Point Set Theory*[2] hold true in S.

DEFINITION. *Suppose that K is a nondegenerate subset of a continuous curve M such that* (1) *if A and B are distinct points of K, there exists a simple closed curve lying in M and containing $A+B$, and* (2) *if X is a point of $M-K$, there is some point O of K such that no simple closed curve lying in M contains both X and O. The set K is said to be a "cyclic nucleus"*[3] *of M.*

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[1] Presented to the Society, December 28, 1939.

[2] American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. Hereinafter, this book will be referred to as *Foundations*. The reader is referred to *Foundations* for the definition of terms used, but not specifically defined, here.

[3] A continuous curve is defined to be a connected im kleinen continuum. It need *not* be locally compact. It is easy to see with the help of Theorems 118 and 120 in Chapter I and the arguments for Theorems 6 and 7 in Chapter II of *Foundations* that if a nondegenerate continuous curve M is regarded as a space and the term "region" is interpreted to mean a connected open subset of M, then with respect to this interpretation of "point" and "region," Axioms 0, 1, and 2 of *Foundations* hold true in M and "limit point" is invariant under this change. Hence, it is possible to apply certain theorems found in Chapter II of *Foundations* and elsewhere to continuous curves. For example, any two points of a connected open subset D of a continuous curve are the extremities of an arc lying wholly in D.

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THEOREM 1. *If K is a cyclic nucleus of a continuous curve M, and AB is an arc which lies in M and whose end points belong to K, then AB is a subset*[4] *of K.*

PROOF. Suppose that O is any point of K and that X is a point of AB. There exist in M simple closed curves J_A and J_B containing $O+A$ and $O+B$ respectively. Let H denote $AB+J_A+J_B$. It is easy to see that H is a compact continuous curve and that no point separates O from X in H. By Theorem 40 on page 124 of *Foundations*, H contains a simple closed curve containing $O+X$. Hence X belongs to K.

THEOREM 2. *If K is a cyclic nucleus of a continuous curve M, and AB is an arc which lies in M and whose end points belong to $\overline{K}$, then $AB-(A+B)$ is a subset of K.*

PROOF. Suppose that X is a point of $AB-(A+B)$. There exist two mutually exclusive connected open subsets D_A and D_B of M which contain A and B respectively such that neither contains X. Since both D_A and D_B contain points of K, D_A+AB+D_B contains an arc $A'B'$ whose end points A' and B' belong to $K\cdot D_A$ and $K\cdot D_B$ respectively. But $A'B'$ must contain X. Hence, by Theorem 1, X belongs to K.

THEOREM 3. *If K is a cyclic nucleus of a continuous curve M, and J is a simple closed curve lying in M and containing more than one point of $\overline{K}$, then J is a subset of K.*

Theorem 3 follows at once from Theorem 2.

THEOREM 4. *If K is a cyclic nucleus of a continuous curve M, then neither K nor $\overline{K}$ contains a cut point of itself.*

Theorem 4 may be established with the help of Theorem 3.

THEOREM 5. *If K_1 and K_2 are cyclic nucleae of a continuous curve M, and $\overline{K}_1$ and $\overline{K}_2$ are identical, then K_1 and K_2 are identical.*

PROOF. If X is a point of K_1, X is a subset of a simple closed curve J lying in K_1. Hence J is a subset of $\overline{K}_2$. By Theorem 3, J is a subset of K_2, and X belongs to K_2. Likewise, any point of K_2 is a point of K_1.

[4] A subset K of a continuous curve M which contains every arc in M whose end points belong to K has been called a *basic set* of M. See W. L. Ayres, *Concerning the arc-curves and basic sets of a continuous curve*, Transactions of this Society, vol. 30 (1928), pp. 567–578, and *Concerning the arc-curves and basic sets of a continuous curve, second paper*, Transactions of this Society, vol. 31 (1929), pp. 595–612.

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THEOREM 6. *If K is a cyclic nucleus of a continuous curve M, and C is a component of $M-\overline{K}$, then C has at most one boundary point in $\overline{K}$.*[5]

PROOF. It follows from Theorem 2 on page 89 of *Foundations* that C is a connected open subset of M. Suppose that C has two boundary points in $\overline{K}$. Using the theorem just referred to, it is easy to see that M contains an arc AB such that (1) $AB-(A+B)$ is a subset of C and (2) both A and B belong to $\overline{K}$. By Theorem 2, $AB-(A+B)$ belongs to K, which is a contradiction.

THEOREM 7. *If K is a cyclic nucleus of a continuous curve M, and X is a point of $M-K$, then there is at most one point O of K such that some simple closed curve in M contains $O+X$.*

PROOF. If X is not a point of $\overline{K}$, the conclusion of Theorem 7 follows from Theorem 6. If X is a point of $\overline{K}$ and a simple closed curve in M contains X and a point O of K, then by Theorem 3, X belongs to K, which is a contradiction.

THEOREM 8. *If K is a cyclic nucleus of a continuous curve M, and D is a connected open subset of M containing a point of K, then both $D\cdot K$ and $D\cdot\overline{K}$ are connected.*[6]

PROOF. Suppose that either $D\cdot K$ or $D\cdot\overline{K}$ is not connected. Then it is the sum of two mutually separate sets, K_1 and K_2. Let AB denote an arc in D from a point of K_1 to a point of K_2. By Theorem 2, $AB-(A+B)$ is a subset of K. This involves a contradiction.

THEOREM 9. *If K is a cyclic nucleus of a continuous curve M, $\overline{K}$ is a nondegenerate continuous curve which contains no cut point of itself.*

Theorem 9 may be proved with the help of Theorems 4 and 8.

THEOREM 10. *If K_1 and K_2 are cyclic nucleae of a continuous curve M, then either K_1 and K_2 are identical or $\overline{K}_1$ and $\overline{K}_2$ have at most one point in common.*

PROOF. Suppose that $\overline{K}_1$ contains two points of $\overline{K}_2$. Then it is evident from Theorems 4 and 6 that $\overline{K}_1$ is a subset of $\overline{K}_2$. Conversely, $\overline{K}_2$ is likewise a subset of $\overline{K}_1$. Hence $\overline{K}_1$ and $\overline{K}_2$ are identical. By Theorem 5, K_1 and K_2 are identical.

[5] Cf. G. T. Whyburn, *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), p. 33, Theorem 2.

[6] If the word "open" is omitted from the statement of Theorem 8, the resulting proposition is false. Cf. G. T. Whyburn, *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), p. 191, Theorem 30.

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THEOREM 11. *If J is a simple closed curve lying in a continuous curve M, then there exists one and only one cyclic nucleus of M containing*[7] *J.*

PROOF. Let G denote the collection of all subsets k of M such that (1) k contains J and (2) every two points of k belong to a simple closed curve in M. Let K denote G^* (that is, the sum of the elements of G) and let A and B denote two distinct points of J. If X and Y are two distinct points of K, there exist in M simple closed curves J_{AX}, J_{BX}, J_{AY}, and J_{BY} which contain $A+X$, $B+X$, $A+Y$, and $B+Y$ respectively. Then $J_{AX}+J_{BX}+J_{AY}+J_{BY}$ is a compact continuous curve no point of which separates X from Y; hence it contains a simple closed curve C containing $X+Y$. Consequently K is an element of G.

Suppose that K is not a cyclic nucleus of M. Then $M-K$ contains a point X such that if O is a point of K, M contains a simple closed curve J_{OX} containing both O and X. Again let A and B denote distinct points of J, and let J_{AX} and J_{BX} denote simple closed curves lying in M and containing $A+X$ and $B+X$ respectively. Then $k=J+J_{AX}+J_{BX}$ is a compact continuous curve containing no cut point of itself, and every two of its points belong to a simple closed curve in k which is a subset of M. Hence k is an element of G. But this involves a contradiction. Therefore K is a cyclic nucleus of M which contains J. By Theorem 10, no other cyclic nucleus of M contains J.

THEOREM 12. *In order that a set K be a cyclic nucleus of a continuous curve M, it is necessary and sufficient that K consist of two points A and B which belong to a simple closed curve in M together with all other points X such that* (1) *some simple closed curve lying in M contains $A+X$ and* (2) *some simple closed curve lying in M contains $B+X$.*

PROOF. It is easily seen from Theorem 7 and the definition of a cyclic nucleus of a continuous curve that the condition is necessary. The condition is also sufficient. For let J denote a simple closed curve in M containing $A+B$ and let H denote the cyclic nucleus of M which contains J. Then H is a subset of K. But it is clear from Theorem 7 that K is a subset of H. Hence K is a cyclic nucleus of M.

THEOREM 13. *If A, B and X are three points of a cyclic nucleus K*

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[7] Cf. G. T. Whyburn, *Cyclicly connected continuous curves*, loc. cit., p. 34, Theorem 3.

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of a continuous curve M, then there exists in K a simple closed curve containing $A+B+X$.

PROOF. Let J denote a simple closed curve in K containing $A+B$. If X does not belong to J, let J_A and J_B denote simple closed curves in M containing $A+X$ and $B+X$ respectively. Suppose that neither J_A nor J_B intersects J in more than one point. Let AB denote an arc of J from A to B. The set $AB+J_A+J_B$ is a compact continuous curve no point of which separates A from B. By Theorem 40 of page 124 of *Foundations*, $AB+J_A+J_B$ contains a simple closed curve C containing $A+B$. Evidently C intersects each of the curves J_A and J_B in more than one point. So in any case there exist a simple closed curve J_1 in M containing $A+B$ and a simple closed curve J_2 in M containing X such that J_2 intersects J_1 in more than one point. Let T denote the component of $J_2-J_1\cdot J_2$ which contains X. Since T is an arc segment, it is easy to see that J_1+T contains a simple closed curve J_3 containing $A+B+X$. By Theorem 3, J_3 is a subset of K.

THEOREM 14. *In order that a set K of more than two points be a cyclic nucleus of a continuous curve M, it is necessary and sufficient that K consist of two points A and B together with all points X such that some simple closed curve lying in M contains $A+B+X$.*

Theorem 14 follows from Theorems 12 and 13.

DEFINITIONS. *A nondegenerate continuous curve M is said to be "almost cyclicly connected" provided that, if A and B are distinct points of M, and R_A and R_B are regions containing A and B respectively, then M contains a simple closed curve containing both a point of R_A and a point of R_B. A subset H of a continuous curve M is said to be an "almost cyclic element" of M provided that* (1) *H is either a cut point or an end point of M or* (2) *H is a nondegenerate almost cyclicly connected continuous curve which is a subset of M but which is not a proper subset of any other almost cyclicly connected continuous curve which is a subset*[8] *of M.*

THEOREM 15. *If K is a cyclic nucleus of a continuous curve M, then $\overline{K}$ is a nondegenerate almost cyclic element of M.*

PROOF. By Theorem 9, $\overline{K}$ is a nondegenerate continuous curve. Since every two points of K lie together in a simple closed curve which is a subset of K, it is evident that $\overline{K}$ is almost cyclicly con-

[8] Cf. G. T. Whyburn, *Concerning the structure of a continuous curve*, loc. cit., p. 167.

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nected. Furthermore, it is clear from Theorem 6, that $\overline{K}$ is not a proper subset of an almost cyclicly connected continuous curve lying in M. Hence $\overline{K}$ is an almost cyclic element of M.

THEOREM 16. *If H is a nondegenerate almost cyclic element of a continuous curve M, then M contains one and only one cyclic nucleus K such that $\overline{K}=H$.*

PROOF. Let J denote a simple closed curve lying in H. By Theorem 11, M contains one and only one cyclic nucleus K containing J. If H is not a subset of $\overline{K}$, then it follows from Theorem 6 that H contains a cut point of itself. This is impossible. Hence H is a subset of $\overline{K}$. But by Theorem 15, $\overline{K}$ is an almost cyclicly connected continuous curve. Consequently, H is not a proper subset of $\overline{K}$.

THEOREM 17. *No two almost cyclic elements of a continuous curve M have more than one point in common.*

THEOREM 18. *If J is a simple closed curve lying in a continuous curve M, then one and only one almost cyclic element of M contains the curve J.*

DEFINITIONS. *Suppose that P is a point of a continuum M and there do not exist two points A and B of M such that* (1) *P separates A from B in M and* (2) *P is the only point of M which separates A from B in M. Let K denote the set of all points X of M such that no point separates P from X in M. Then K will be called a "simple link of M" and P will be called a "proper point of M."*[9]

THEOREM 19. *Every nondegenerate simple link of a continuous curve M is an almost cyclic element*[10] *of M.*

PROOF. Suppose that K is a nondegenerate simple link of M. Let A and B denote two distinct points of K and let AXB denote an arc in M from A to B containing a point X distinct from A and B. The arc AXB belongs to K, and X does not separate A from B in K. Hence there exists in $M-X$ an arc $A'B'$ having only its end points A' and B' in AXB. Let J denote the simple closed curve contained in $AXB+A'+B'$. Obviously J is a subset of K. By Theorem 18, M contains one and only one almost cyclic element H containing J. Since no two points of K are separated in M by any point of K, it follows from Theorems 6 and 16 that K is a subset of H. On the other hand, K contains a point P together with all other points X of M such that

[9] See pages 63 and 72 of *Foundations.*

[10] Cf. Theorem 68 on page 148 of *Foundations.*

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X is not separated in M from P by a point of M. Since H contains P and no cut point of itself, it is a subset of K. Consequently K is identical with H.

THEOREM 20. *Every simple link of a continuous curve is itself a continuous curve.*[11]

THEOREM 21. *If a nondegenerate almost cyclic element H of a continuous curve M contains a proper point of M, then H is a simple link*[12] *of M.*

PROOF. If H contains a proper point P of M, then by Theorem 94 on page 68 of *Foundations* M contains one and only one simple link K containing P. Evidently every point of H belongs to K. It follows from Theorems 17 and 19 that H and K are identical.

EXAMPLE. *An almost cyclic element of a continuous curve M need not contain a proper point of M.* Suppose that E is an euclidean 3-space. Let H denote all points $(X, Y, 0)$ of E such that $X^2+Y^2=1$ and let W denote all points (X, Y, Z) of E such that $X^2+Y^2>1$. Now let M denote $H+W$ and define "region" in M as follows: (1) if R is a region in E containing a point of H, then $R\cdot M$ shall be called a "region" in M and (2) if (X, Y, Z) is a point of W and R is a circular region of the plane $Yx-Xy=0$ containing (X, Y, Z) such that $\overline{R}$ contains no point of H, then $R\cdot M$ shall be called a "region" in M. The space M satisfies Axioms 0–2 of *Foundations*. Consequently M is a continuous curve lying in a complete Moore space. It is clear that H is an almost cyclic element of M which contains no proper point of M.

With the theorems and proofs of the preceding pages in mind the reader will find it possible to establish many results for *almost cyclic* elements of a continuous curve in a complete Moore space analogous to those in the literature[13] for *cyclic* elements of a continuous curve in an euclidean n-space by slight changes in the original arguments. For certain results, such as those concerning the number of nondegenerate cyclic elements, it is obviously necessary in complete Moore spaces to assume that the continuous curves involved are separable. One theorem of considerable interest is as follows: *Suppose that M is a continuous curve in a complete Moore space. If the almost cyclic elements of M are regarded as "points" and two such "points" p and q are*

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[11] Cf. Theorem 66 on page 147 of *Foundations.*

[12] Cf. Theorem 69 on page 149 of *Foundations.*

[13] Especially those of the references which have been given in the footnotes of this paper.

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regarded as contiguous if and only if one of the two continua p and q is a point of the other, then the set of all such "points" is an acyclic continuous curve.[14]

2. **Spaces in which the Jordan curve theorem holds true.** The next two theorems are obtained by a simple application of the preceding results to spaces in which the Jordan curve theorem holds true.

THEOREM 22. *Suppose that S is a space satisfying Axioms* 0, 1, 2, *and* 4 *of Foundations and that H is a nondegenerate almost cyclic element of S. If H is regarded as a space and the term "region" is interpreted to mean a connected open subset of H, then, with respect to this interpretation of "point" and "region," Axioms* 0–4 *of Foundations are satisfied and "limit point" is invariant under this change.*[15]

PROOF. By Theorem 9 on page 96 of *Foundations*, Axioms 0–2 are satisfied in H and "limit point" is invariant. It follows from Theorems 4 and 16 that Axiom 3 is satisfied. If J is a simple closed curve lying in H, then $S-J$ is the sum of two connected domains, E and I, each having J for its boundary. It follows from Theorems 8 and 16 that $H\cdot E$ and $H\cdot I$, if not vacuous, are connected open subsets of H. If some point P of J is not a limit point of $H\cdot E$, then some segment T of J contains P but no point of $H\cdot \overline{E}$. There exists an arc segment W lying in E whose end points lie in T. By Theorems 2 and 16, W is a subset of H. Hence W is in $H\cdot E$, and the end points of W belong to both $H\cdot \overline{E}$ and T, which is a contradiction. Consequently, every point of J is a limit point of $H\cdot E$. It is now evident that J is the boundary with respect to H of $H\cdot E$. Likewise J is the boundary with respect to H of $H\cdot I$. Hence Axiom 4 holds true in H.

THEOREM 23. *Suppose that S is the set of all points and that Axioms* 0, 1, 2, *and* 4 *of Foundations hold true. Then a nondegenerate simple link of S is a nondegenerate almost cyclic element of S and conversely.*

PROOF. It is obvious from Axiom 4 that no point of a simple closed curve is a cut point of S. Hence every point of a simple closed curve

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[14] Cf. R. L. Moore, *Fundamental theorems concerning points sets*, the Rice Institute Pamphlet, vol. 23 (1936), pp. 1–74; p. 74 in particular.

[15] AXIOM 2. *If P is a point of a region R, there exists a nondegenerate connected domain containing P and lying wholly in R.*

AXIOM 3. *If O is a point, $S-O$ is connected.*

AXIOM 4. *If J is a simple closed curve, $S-J$ is the sum of two mutually separated connected point sets such that J is the boundary of each of them.* (The Jordan curve theorem.)

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is a proper point of S. The conclusion of Theorem 23 now follows immediately from Theorems 19 and 21.

I wish to point out that Theorems 22 and 23 shed some light on the role played by Axiom 3 in the sequence of axioms in *Foundations*. A space satisfying Axioms 0–4 of *Foundations* is identical with its one almost cyclic element. Hence, by assuming Axiom 3 in addition to Axioms 0, 1, 2, and 4 one has merely confined one's investigation to a single almost cyclic element of the space. In fact with the preceding theorems in mind it is easy to see that many of the theorems in the literature which hold true in spaces satisfying Axioms 0–4 also hold true in spaces satisfying only Axioms 0, 1, 2, and 4. This is true, for instance, of all the theorems in Chapter III of *Foundations* except Theorems 0, 2, 21, 23, and 24.

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