ON REARRANGEMENTS OF SERIES

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1. Introduction. Let $E$ denote the metric space in which a point $x$ is a permutation $x_1, x_2, x_3, \ldots$ of the positive integers and the distance $(x,y)$ between two points $x \equiv \{x_1, x_2, \ldots\}$ and $y \equiv \{y_1, y_2, \ldots\}$ of $E$ is given by the Fréchet formula

$$ (x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}. $$

The space $E$ is of the second category (Theorem 2).

Let $c_1 + c_2 + \cdots$ be a convergent series of real terms for which $\sum |c_n| = \infty$. To simplify typography, we write $c(n)$ for $c_n$. To each $x \in E$ corresponds a rearrangement $c(x_1) + c(x_2) + \cdots$ of the series $\sum c_n$. By a well known theorem of Riemann, $x \in E$ exists such that $c(x_1) + c(x_2) + \cdots$ converges to a preassigned number, diverges to $+\infty$ or to $-\infty$, or oscillates in a preassigned fashion.

The set $A$ of $x \in E$ for which $c(x_1) + c(x_2) + \cdots$ converges is therefore a proper subset of $E$, and M. Kac has proposed the problem of determining whether $E - A$ is of the second category. The following theorem shows not only that $A$ is of the first category (and hence that $E - A$ is of the second category) but also that the set of $x \in E$ for which the series $c(x_1) + c(x_2) + \cdots$ has unilaterally bounded partial sums is of the first category.

Theorem 1. For each $x \in E$ except those belonging to a set of the first category,

$$ \lim \inf_{N \to \infty} \sum_{n=1}^{N} c(x_n) = -\infty, \quad \lim \sup_{N \to \infty} \sum_{n=1}^{N} c(x_n) = \infty. $$

2. Proof of Theorem 1. The fact that the "coordinates" $x_n$ and $y_n$ of two points $x$ and $y$ of $E$ are integers implies roughly that, if $N$ is large, then $x_n = y_n$ for $n = 1, 2, \cdots, N$ if and only if $(x, y)$ is near 0. To make this precise, let $x \in E$, $r > 0$, and let $S(x, r)$ denote the set of points $y$ such that $(x, y) < r$, so that $S(x, r)$ is an open sphere with center at $x$ and radius $r$. It is easy to show that if $x$ and $y$ are two points of $E$ such that $y \in S(x, 2^{-N-1})$ then $x_n = y_n$ when $n = 1, 2, \cdots, N$; and that if $x$ and $y$ are such that $x_n = y_n$ when $n = 1, 2, \cdots, N$ then $y \in S(x, 2^{-N})$.

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To prove Theorem 1, let \( B \) denote the set of \( x \in E \) for which

\[
\limsup_{N \to \infty} \sum_{n=1}^{N} c(x_n) < \infty;
\]

and, for each \( h > 0 \), let \( B_h \) denote the set of \( x \in E \) for which

\[
\text{l.u.b.} \sum_{n=1}^{N} c(x_n) < h.
\]

Then

\[
B = B_1 + B_2 + B_3 + \cdots.
\]

We show that \( B \) is the first category by showing that \( B_h \) is nondense for each \( h > 0 \). Suppose \( h > 0 \) exists such that the closure \( \overline{B}_h \) of \( B_h \) contains a sphere \( S' \) with center at \( x' = \{ x'_1, x'_2, \ldots \} \) and radius \( r > 0 \). Choose \( m \) so great that \( 2^{-m-1} + 2^{-m-2} + \cdots < r/2 \). Let \( x''_n = x'_n \) when \( 1 \leq n \leq m \); and define \( x''_n \) for \( n > m \) in such a way that \( \sum c(x''_n) \) diverges to \( +\infty \). Then \( (x', x'') < r/2 \) so that \( x'' \in S' \). Choose an index \( q \) such that

\[
c(x''_1) + c(x''_2) + \cdots + c(x''_q) > h,
\]

and then choose \( \delta > 0 \) such that \( x_k = y_k \) for \( k = 1, 2, \ldots, q \) whenever \( x, y \in E \) and \( (x, y) < \delta \).

If \( x \) is a point within the sphere \( S'' \) with center at \( x'' \) and radius \( \delta \) (that is, if \( (x, x'') < \delta \)), then \( c(x_1) + c(x_2) + \cdots + c(x_q) > h \) and \( x \) is not in \( B_h \). Thus \( B_h \) contains no point of \( S'' \) and consequently \( \overline{B}_h \) does not contain \( x'' \). This contradicts the assumption that \( \overline{B}_h \) contains \( S' \), and hence proves that \( B_h \) is nondense and \( B \) is of the first category. Similar considerations show that the set \( C \) of \( x \in E \) for which \( c(x_1) + \cdots + c(x_N) \) has inferior limit greater than \( -\infty \) is of the first category. Since the union of two sets \( B \) and \( C \) of the first category is itself of the first category, Theorem 1 is established.

If \( z_1 + z_2 + \cdots \) is a convergent series of complex terms for which \( \sum |z_n| = \infty \), it is easy to apply our theorem to the series of real and imaginary parts of \( z_n \) to show that the set of \( x \in E \) for which \( z(x_1) + z(x_2) + \cdots \) has bounded partial sums is a set of the first category.

3. The space \( E \). In this section we obtain some properties of \( E \) and prove the following result.

**Theorem 2.** The space \( E \) is of the second category at each of its points.
That the space $E$ is not complete was pointed out to the author by Professor L. M. Graves. In fact if $x^{(n)}$ is the point

$$x^{(n)} = \{2, 3, \ldots, n - 1, n, 1, n + 1, n + 2, \ldots\}$$

of $E$, then $x^{(n)}$ is a Cauchy sequence in $E$ which does not converge to a point of $E$. If $\mathcal{E}$ is the space in which a point is a sequence of positive integers not necessarily a permutation of all positive integers, and the distance between two points of $\mathcal{E}$ is given by the Fréchet formula, then $\mathcal{E}$ is complete and $E$ is a subspace of $\mathcal{E}$. It is easy to show that the closure of $E$ in $\mathcal{E}$ is the space $\mathcal{E}_1$ in which a point is a sequence of positive integers containing each positive integer at most once, and hence that $\mathcal{E}_1$ is the least complete subspace of $\mathcal{E}$ which contains $E$. For example, $\{2, 4, 6, 8, \ldots\}$ is a point of $\mathcal{E}_1$ which is not a point of $E$.

If $\mathcal{E}_x\{x_n = k\}$ denotes, for each $n, k = 1, 2, \ldots$, the set of all $x \in \mathcal{E}$ for which $x_n = k$, then

$$\mathcal{E}_2 = \prod_{k=1}^{\infty} \sum_{n=1}^{\infty} \mathcal{E}_x\{x_n = k\}$$

is the subset of $\mathcal{E}$ in which a point is a sequence containing each positive integer at least once. Since $\mathcal{E}_x\{x_n = k\}$ is an open subset of $\mathcal{E}$ for each $n, k = 1, 2, \ldots$, $\mathcal{E}_2$ is the intersection of a countable set of open sets (that is, $\mathcal{E}_2$ is a $G_\delta$) in $\mathcal{E}$. Since $\mathcal{E}_1$ is a closed subset of $\mathcal{E}$ and $E = \mathcal{E}_1 \mathcal{E}_2$, it follows that $E$ is a $G_\delta$ in the complete space $\mathcal{E}$.

Therefore, by a fundamental theorem whose proof is an easy extension of the familiar proof that a complete metric space is of the second category, $E$ is of the second category at each of its points and Theorem 2 is proved.

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