ABELIAN GROUPS THAT ARE DIRECT SUMMANDS OF EVERY CONTAINING ABELIAN GROUP

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It is a well known theorem that an abelian group $G$ satisfying $G = nG$ for every positive integer $n$ is a direct summand of every abelian group $H$ which contains $G$ as a subgroup. It is the object of this note to generalize this theorem to abelian groups admitting a ring of operators, and to show that the corresponding conditions are not only sufficient but are at the same time necessary. Finally we show that every abelian group admitting a ring of operators may be imbedded in a group with the above mentioned properties; and it is possible to choose this “completion” of the given group in such a way that it is isomorphic to a subgroup of every other completion.

Our investigation is concerned with abelian groups admitting a ring of operators. A ring $R$ is an abelian group with regard to addition, its multiplication is associative, and the two operations are connected by the distributive laws. As the multiplication in $R$ need not be commutative, we ought to distinguish left-, right- and two-sided ideals. Since, however, only left-ideals will occur in the future, we may use the term “ideals” without fear of confusion. Thus an ideal in $R$ is a non-vacuous set $M$ of elements in $R$ with the property:

If $m', m''$ are elements in $M$, and if $r', r''$ are elements in $R$, then $r'm' \pm r''m''$ is an element in $M$.

An abelian group $G$ whose composition is written as addition is said to admit the elements in the ring $R$ as operators (or shorter: $G$ is an abelian group over $R$), if with every element $r$ in $R$ and $g$ in $G$ is connected their uniquely determined product $rg$ so that this product is an element in $G$ and so that this multiplication satisfies the associative and distributive laws. If $G$ is an abelian group over $R$, then its subgroups $M$ are characterized by the same property as the ideals $M$ in $R$.

We assume finally the existence of an element 1 in $R$ so that $g = 1g$ for every $g$ in $G$ and $r \cdot 1 = 1 \cdot r = r$ for every $r$ in $R$.

If $x$ is any element in the abelian group $G$ over $R$, then its order $N(x)$ consists of all the elements $r$ in $R$ which satisfy $rx = 0$. One verifies that every order $N(x)$ is an ideal in $R$, and that $N(x) = R$ if, and only if, $x = 0$.

If $M$ is an ideal in $R$, and if $x$ is an element in $G$, then a subgroup

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1 Presented to the Society, February 24, 1940.
of $G$ is formed by the elements $mx$ for $m$ in $M$; and this subgroup may be denoted by $Mx$. (It is a subgroup of the cyclic group generated by $x$.) The correspondence between the element $m$ in $M$ and the element $mx$ in $Mx$ is a special case of the homomorphisms of $M$ into $G$. Here a homomorphism $\phi$ of the ideal $M$ in $R$ into the abelian group $G$ over $R$ is defined as a single-valued function $m^\phi$ of the elements $m$ in $M$ with values in $G$ which satisfies

$$(r'm' \pm r''m'')^\phi = r'(m')^\phi \pm r''(m'')^\phi$$

for $m', m''$ in $M$ and $r', r''$ in $R$.

We are now ready to state and prove our main result.

**Theorem 1.** The following two properties of an abelian group $G$ over the ring $R$ are each a consequence of the other.

(a) If $G$ is a subgroup of the abelian group $H$ over $R$, then $G$ is a direct summand of $H$.

(b) To every ideal $M$ in $R$ and to every homomorphism $\phi$ of $M$ into $G$ there exists some element $v$ in $G$ so that $m^\phi = mv$ for every $m$ in $M$.

**Proof.** Assume first that (a) is satisfied by $G$. If $M$ is an ideal in $R$, and if $\phi$ is a homomorphism of $M$ into $G$, then there exists one and essentially only one group $H$ over $R$ which is generated in adjoining to $G$ an element $h$, subject to the relations

$$mh = m^\phi$$

for every $m$ in $M$.

It is a consequence of (a) that $H$ is the direct sum of $G$ and of a suitable subgroup $K$ of $H$ so that every element in $H$ may be represented in one and only one way in the form: $g+k$ for $g$ in $G$ and $k$ in $K$. This applies in particular to the element $h$ so that $h=v+w$ for

2 The following is a remark by the referee: "It is perhaps of some interest to observe that Theorem 1 contains a generalization of the theorem that every representation of a semisimple algebra is fully reducible. Indeed, how does one characterize those rings $R$ such that every abelian group $G$ admitting $R$ as an operator ring has the property (a)? The answer is that every left-ideal in $R$ must be generated by an idempotent element, and this is equivalent to saying that $R$ is semisimple (both chain conditions and no radical)."

"The sufficiency of this condition is proved by showing that $G$ has the property (b). Let $m \rightarrow m^\phi$ be any homomorphism of a left-ideal $M$ of $R$ into $G$. Since $M = Re$ with $e^2 = e$, then $me = m$ for every $m$ in $M$, and if we take $v = e^\phi$ we have $m^\phi = (me)^\phi = me^\phi = mv$.

"The necessity is proved by observing that every left-ideal $M$ of $R$ is itself an abelian group over $R$, and is a subgroup of $R$. Hence to each $M$ there exists a complementary left-ideal $N$. From $1 = e + e'$ with unique $e$ in $M$, $e'$ in $N$, one concludes in the usual way that $e^2 = e$ and $M = Re$."

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uniquely determined elements $v$ and $w$ in $G$ and $K$ respectively. Then $m^* = mh = mv + mw$ is for every element $m$ in $M$ an element in $G$, and so is $mv$ and $m^* - mw = mw$. Since $mw$ is an element in $K$, it follows that $mw = 0$ or $m^* = mv$ for every $m$ in $M$; and this shows that (b) is a consequence of (a).

Assume now conversely that (b) is satisfied by the abelian group $G$ over $R$, and that $G$ is a subgroup of the abelian group $H$ over $R$. Then there exists a greatest subgroup $K$ of $H$ whose meet with $G$ is 0. The subgroup $S$ of $H$ which is generated by $G$ and $K$ is their direct sum; and hence it suffices to prove that $H = S = G + K$.

If $S \neq H$, then there exists in $H$ an element $w$ that is not contained in $S$. The coset $W = S + w$ is an element in the quotient-group $H/S$ and its order $N(W)$ is an ideal in $R$. If $m$ is any element in $N(W)$, then $mw$ is an element in $S$; and it follows from the construction of $S$ that there exist uniquely determined elements $g(m)$ and $k(m)$ in $G$ and $K$ respectively so that $mw = g(m) + k(m)$ for $m$ in $N(W)$. Thus a homomorphism of $N(W)$ into $G$ is defined in mapping the element $m$ in $N(W)$ upon the element $g(m)$ in $G$. There exists therefore by condition (b) an element $v$ in $G$ so that $mv = g(m)$ for every $m$ in $N(W)$. The element $w' = w - v$ consequently satisfies $S + w' = S + w = W$ and

$$mw' = k(m)$$

for every element $m$ in $N(W)$. Since $w$ is not an element in $S$, neither is $w'$. Since $K$ is a greatest subgroup of $H$ whose meet with $G$ is 0, and since $w'$ is not an element in $S$ and therefore not in $K$, adjoining $w'$ to $K$ generates a subgroup whose meet with $G$ is different from 0. Hence there exists an element $k$ in $K$, and an element $r$ in $R$ so that $k + rw' = g \neq 0$

is an element in $G$. Since $rw' = g - k$ is an element in $S$, it follows that $r$ is an element in $N(W)$ so that $g = k + k(r)$ is an element in the meet of $G$ and $K$. This contradicts, however, the construction of $K$ and the choice of $g$. Our hypothesis $S \neq H$ has thus led us to a contradiction; and this completes the proof.

If $M$ is an ideal in the ring $R$, and if $G$ is an abelian group over $R$, then $G$ is termed $M$-complete, if there exists to every homomorphism $\phi$ of $M$ into $G$ an element $v$ in $G$ so that $m^* = mv$ for every $m$ in $M$. In
this terminology, condition (b) of Theorem 1 states that $G$ is $M$-complete for every ideal $M$ in $R$. It is our object to characterize the $M$-complete groups, provided $M$ is a principal ideal. For this end we need several notations.

If $M$ is an ideal in $R$, then denote by $G_M$ the set of all the elements $g$ in $G$ which satisfy $mg = 0$ for every $m$ in $M$. Note that $G_M$ need not be a subgroup of the abelian group $G$ over $R$, though it is closed with regard to addition and subtraction.

If $p$ is any element in $R$, then $pG$ consists of all the elements $pg$ for $g$ in $G$. Note again that $pG$ need not be a subgroup of the abelian group $G$ over $R$.

The principal ideal in $R$, generated by the element $p$ in $R$, consists of all the elements $rp$ for $r$ in $R$ and may therefore be denoted by $Rp$; and $N(p)$ consists of all the elements $r$ in $R$ so that $rp = 0$. $N(p)$ is clearly an ideal in $R$.

**Theorem 2.** The abelian group $G$ over $R$ is $Rp$-complete if and only if $G_N(p) \subseteq pG$.

**Proof.** Suppose first that $G$ is $Rp$-complete. If $x$ is an element in $G_N(p)$, then $r'p = r''p$ implies $r'x = r''x$, since the first equation is equivalent to the fact that $r' - r''$ is an element in $N(p)$. Thus a homomorphism of $Rp$ into $G$ is defined in mapping $rp$ upon $rx$. Since $G$ is $Rp$-complete, there exists an element $v$ in $G$ so that $rpv = rx$ for every $r$ in $R$. This implies in particular that $pv = x$, that is, our condition is necessary.

Suppose conversely that our condition be satisfied by $G$. If $\phi$ is a homomorphism of $Rp$ into $G$, then $p\phi$ is an element in $G_{N(p)}$, since we have $r(p\phi) = (rp)\phi = 0$ for elements $r$ in $N(p)$. Hence there exists an element $g$ in $G$ so that $p\phi = pg$; and clearly $(rp)\phi = r(p\phi) = r(pg) = (rp)g$ for every $r$ in $R$ so that $G$ is $Rp$-complete.

**Corollary 1.** If $p$ is an element in $R$ so that $N(p) = 0$, then $G = pG$ is a necessary and sufficient condition for $Rp$-completeness of the abelian group $G$ over $R$.

This is a consequence of Theorem 2, since $G_0 = G$.

**Corollary 2.** If $N(p) = 0$ for every element $p \neq 0$ in the ring $R$, and if every ideal in $R$ is a principal ideal $Rp$, then $G = pG$ for every $p \neq 0$ in $R$ is a necessary and sufficient condition for the abelian group $G$ over $R$ to be a direct summand of every abelian group $H$ over $R$ which contains $G$ as a subgroup.

This is an obvious consequence of Theorem 1 and Corollary 1.
If, in particular, $R$ consists of the rational integers, then the hypotheses of Corollary 2 are satisfied. In this case the sufficiency of the condition of the Corollary 2 has been known for a long time.³

An abelian group $G$ over the ring $R$ is termed complete, if it is $M$-complete for every ideal $M$ in $R$. Thus the complete groups are just the groups satisfying the properties (a) and (b) of Theorem 1.

**Theorem 3.** Every abelian group over the ring $R$ is a subgroup of a complete abelian group over the ring $R$.

**Proof.** If $G$ is an abelian group over the ring $R$, $M$ an ideal in $R$, and $\phi$ a homomorphism of $M$ into $G$, then there exists an abelian group $H$ over $R$ which contains $G$ as a subgroup and which contains an element $x$ so that $mx = m^*$ for every $m$ in $M$.

By repetition of the construction of the preceding paragraph one may show that if $G$ is an abelian group over the ring $R$, then there exists an abelian group $G'$ over the ring $R$ which contains $G$ as a subgroup and which satisfies the following condition:

(3.1) If $M$ is an ideal in $R$, and if $\phi$ is a homomorphism of $M$ into $G$, then there exists an element $v$ in $G'$ so that $mv = m^*$ for every $m$ in $M$.

Denote now by $\lambda$ an ordinal number which is a limit-ordinal and whose cardinal number is greater than the number of elements in $R$. Then it follows from the second paragraph of the proof that there exists for every ordinal $\nu$ with $0 \leq \nu \leq \lambda$ an abelian group $G^\nu$ over $R$ with the following properties:

(i) $G^\nu = G$;

(ii) $G^\rho \leq G^\mu$ for $\nu < \mu$;

(iii) $G^\nu$ is for limit-ordinals $\nu$ the set of all the elements contained in groups $G^\mu$ for $\mu < \nu$;

(iv) $G_\mu$ and $G_{\mu+1}$ satisfy condition (3.1).

Suppose now that $M$ is an ideal in $R$ and that $\phi$ is a homomorphism of $M$ into $H = G_\lambda$. Then there exists an ordinal $\sigma < \lambda$ so that $G_\sigma$ contains all the elements $m^*$; and there exists therefore an element $v$ in $G_{\sigma+1}$ so that $mv = m^*$ for every $m$. $H$ is therefore complete.

**Theorem 4.** To every subgroup $G$ of the complete abelian group $K$ over the ring $R$ there exists a complete subgroup $G^*$ of $K$ which contains $G$ as a subgroup and which satisfies the following condition:

(E) Every isomorphism of $G$ upon a subgroup of a complete abelian group $H$ over $R$ is induced by an isomorphism of $G^*$ upon a subgroup of $H$.

Proof. If $T$ is an abelian group over the ring $R$, if $M$ is an ideal in $R$, then the homomorphism $\phi$ of $M$ into $T$ is termed reducible in $T$, if there exists an ideal $M'$ in $R$ and a homomorphism $\phi'$ of $M'$ into $T$ so that $M < M'$ and so that $\phi$ and $\phi'$ coincide on $M$. If $\phi$ is not reducible, then it is irreducible in $T$.

(4.1) The abelian group $T$ over the ring $R$ is complete, if there exists to every ideal $M$ in $R$ and to every irreducible homomorphism $\phi$ of $M$ into $T$ an element $v$ in $T$ so that $mv = m^\phi$ for every $m$ in $M$.

To prove this statement let $J$ be an ideal in $R$ and $\gamma$ a homomorphism of $J$ into $T$. Then there exists a greatest ideal $M$ in $R$ so that $J \subseteq M$ and so that $\gamma$ is induced in $J$ by a homomorphism $\phi$ of $M$ into $T$. It is clear that $\phi$ is irreducible in $T$. Hence there exists an element $v$ in $T$ so that $nv = n^\phi$ for every $n$ in $J$, that is, $T$ is complete.

It is a consequence of (4.1) and of the completeness of $K$ that there exists an ascending chain of subgroups $G_\nu$ for $0 \leq \nu \leq \lambda$ with the following properties:

(i) $G = G_0$;
(ii) $G_\nu \leq K$ for $\nu \leq \lambda$;
(iii) $G_{\nu+1}$ is generated by adjoining to $G_\nu$ an element $g_\nu$ with the following properties:

(iii') The homomorphism of $N(G_\nu + g_\nu)$ into $G_\nu$, which is defined by mapping the element $m$ in $N(G_\nu + g_\nu)$ upon the element $mg_\nu$ in $G_\nu$, is irreducible in $G_\nu$.

(iii'') $G_\nu$ does not contain any element $x$ so that $mx = mg_\nu$ for every $m$ in $N(G_\nu + g_\nu)$.

(iv) $G_\nu$ is for limit-ordinals $\nu$ the set of all the elements contained in groups $G_\mu$ for $\mu < \nu$.

(v) $G_\lambda = G^*$ is complete.

We are now going to prove that this subgroup $G^*$ of $K$ satisfies condition (E). Thus assume that $\rho$ is an isomorphism of $G$ upon the subgroup $G' = G^\rho$ of the complete group $H$. We are going to construct subgroups $G'_\nu$ of $H$ and isomorphisms $\rho_\nu$ of $G_\nu$ upon $G'_\nu$ with the following properties:

(1) $G' = G_0^\rho$, $\rho = \rho_0$;
(2) $G'_\nu \leq G'_\mu$ for $\nu \leq \mu$;
(3) $\rho_\nu$ and $\rho_\mu$ coincide on $G_\nu$ for $\nu \leq \mu$.

In order to prove the possibility of this construction it suffices to show the existence of $G'_{\nu+1}$, $\rho_{\nu+1}$ under the hypothesis of the existence of $G'_\nu$, $\rho_\nu$.

This condition (iii'') is not really needed for the proof, though it is convenient for the construction of the chain $G_\nu$.
A homomorphism irreducible in $G'_1$ of $M = N(G_r + g_r)$ into $G'_1$ is defined by mapping the element $m$ in $M$ upon the element $m^* = (mg_{r})^{p_r}$. Since $H$ is complete, there exists an element $h$ in $H$ so that $m^* = mh$ for every $m$ in $M$. If $M' = N(G'_1 + h)$, then it is clear that $M \leq M'$. If $m$ is in $M'$, then $mh$ is an element in $G'_1$. Thus a homomorphism $\gamma$ of $M'$ into $G_r$ is defined by mapping the element $m$ in $M$ upon the element $m^* = (mh)^{p_r}$. If, in particular, $m$ is an element in $M$, then $m^* = mg_r$; and it follows from (iii') that $M = M'$. Suppose now that $g'$ is an element in $G'_r$ and $u$ an element in $R$ so that $g' + uh = 0$. Then $u$ is an element in $M = M'$ and it follows from the above considerations that $-g' = uh = u^{p_r} = (ug_{r})^{p_r}$. Hence there exists one and only one isomorphism $\rho_{r+1}$ of $G_{r+1}$ upon the group $G'_{r+1}$, generated by $G'_r$ and $h$, which isomorphism induces $\rho_r$ in $G_r$, and maps $g_r$ upon $h$.

Thus there exists finally an isomorphism $\rho_\lambda$ of $G^* = G_\lambda$ upon $G'_1$ which induces $\rho$ in $G$; and this completes the proof.

**Corollary.** Assume that $K$ is a smallest complete abelian group over the ring $R$ containing the subgroups $G_i$. Then $G_1$ and $G_2$ are isomorphic if, and only if, there exists an automorphism of $K$ mapping $G_1$ upon $G_2$.

This is an obvious consequence of Theorem 4. It should be noted, however, that the complete group $G^*$, satisfying (E) and containing $G$, whose existence is assured by Theorems 3 and 4, is only "essentially smallest," but need not be "actually smallest."

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