CLOSURE OF PRODUCTS OF FUNCTIONS[1]

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This note presents some natural theorems on the characterizations of certain closed (*or complete*) sets of functions with separable variables. In order to motivate the developments of the paper we treat a simple case first in elaborate detail. The proof is so formulated that it holds with trifling modifications for the more general situations in Theorems 3 and 4. The result in Theorem 5 belongs to a slightly different range of ideas.

Let $s\sim(s_1, \cdots, s_m)$ and $t\sim(t_1, \cdots, t_n)$ here stand for points in the euclidean spaces R_m and R_n. The term "interval" designates the generalized rectangular parallelopipedon open on the left.[2] We shall make use of the intervals $I_s \subset R_m$, $I_t \subset R_n$ and $I_2 = I_s \times I_t \subset R_{n+m}$. We are first interested in $L_2(I)$, the space of complex valued functions of summable square over I. The norm and scalar product are defined as usual by

$$\|f(s, t) - g(s, t)\| = \left[\int_{I_t}\int_{I_s} |f(s, t) - g(s, t)|^2 dI_s dI_t\right]^{1/2}, \tag{1}$$

$$(f(s, t), g(s, t)) = \int_{I_t}\int_{I_s} f(s, t)\bar{g}(s, t) dI_s dI_t, \tag{2}$$

where $\bar{g}(s, t)$ is the conjugate of $g(s, t)$. The subscript I_s or I_t will indicate that the left-hand functionals are on the corresponding intervals.

We shall understand closure of the sequence of functions[3] $\{\phi_\gamma(t)\psi_\mu(s)\}$, $\gamma;\ \mu = 0, 1, \cdots$, to mean that for every $f(s, t) \varepsilon L_2(I_2)$ and arbitrary $\epsilon > 0$ there exists a finite sequence of complex constants $\{\beta_{\gamma\mu}\}$ and integers A and B such that

$$\left\|f(s, t) - \sum_0^A \sum_0^B \beta_{\gamma\mu}\phi_\gamma(t)\psi_\mu(s)\right\| < \epsilon. \tag{3}$$

It is well known that with the adjunction of the scalar product defined in (2), $L_2(I_2)$ is a complex Hilbert space and that closure and completeness are equivalent concepts.

THEOREM 1. *If* $\{\phi_\gamma(t)\psi_\mu(s)\}$, $\gamma, \mu = 0, 1, \cdots$, *is a sequence of com-*

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[1] Presented to the Society, December 2, 1939.

[2] S. Saks, *Theory of the Integral*, English edition, p. 57.

[3] Curly brackets, { }, will always denote sequences.

plex valued functions in $L_2(I_2)$, then a necessary and sufficient[4] *condition for closure is that $\{\phi_\gamma(t)\}$ and $\{\psi_\mu(s)\}$ be closed in the spaces $L_2(I_t)$ and $L_2(I_s)$ respectively.*

We deal with the sufficiency demonstration first. Suppose the denumerable set of all subintervals, with rational end points, of I_t to be ordered according to 0, 1, 2, $\cdots$. We designate by $h_\rho(t)$ the characteristic function[5] of the ρth subinterval divided by its norm. The function $g_\nu(s)$ is similarly defined for the range I_s. Thus

$$\|h_\rho(t)\|_{I_t} = \|g_\nu(s)\|_{I_s} = 1. \tag{4}$$

It is well known that $\{h_\rho(t)g_\nu(s)\}$, ρ, $\nu=0, 1, \cdots$, has the closure property in $L_2(I_2)$. Hence for $f(s, t) \in L_2(I_2)$ and arbitrary $\epsilon>0$ we can find integers M and N and MN complex constants $\{a_{\rho\nu}\}$ such that

$$\left\| f(s, t) - \sum_0^M \sum_0^N a_{\rho\nu} h_\rho(t) g_\nu(s) \right\| < \epsilon/2. \tag{5}$$

Let

$$\delta \leqq \min\left(\frac{\epsilon}{4MN} \max |a_{\rho\nu}|, 1\right). \tag{6}$$

Thus

$$2\delta \sum_0^M \sum_0^N |a_{\rho\nu}| < \epsilon/2. \tag{6.1}$$

In view of the assumed closure properties of $\{\phi_\gamma(t)\}$ and $\{\psi_\mu(s)\}$, integers A and B and complex constants $\{d_\mu^{(\nu)}\}$, $\{e_\gamma^{(\rho)}\}$, $\rho=0, 1, \cdots, M$ and $\nu=0, 1, \cdots, N$, exist which yield the simultaneous inequalities

$$\left\| g_\nu(s) - \sum_{\mu=0}^B d_\mu^{(\nu)} \psi_\mu(s) \right\|_I < \delta/2, \tag{7}$$

$$\left\| h_\rho(t) - \sum_0^A e_\gamma^{(\rho)} \phi_\gamma(t) \right\|_{I_t} < \delta/2. \tag{7.1}$$

Hence

[4] A special case amounting to the assertion of sufficiency, only, for the subspace of $L_2(I_2)$ composed of real continuous functions, when $\{\phi_\gamma(t)\}$ and $\{\psi_\mu(s)\}$ are restricted to be orthogonal sets of functions, has been given by Courant: Courant-Hilbert, *Methoden der mathematischen Physik*, vol. 1, 1st edition, p. 90. Another special sufficiency proof is given in A. Zymund, *Trigonometrical Series*, p. 13.

[5] Saks, loc. cit., p. 6.

$$(7.2)\qquad \left\| \sum_0^B d_\mu^{(\nu)} \psi_\mu(s) \right\|_{I_s} \leqq \|g_\nu(s)\|_{I_t} + \left\| g_\nu(s) - \sum_0^B d_\mu^{(\nu)} \psi_\mu(s) \right\|_{I_s} \leqq 2.$$

Let $\beta_{\gamma\mu} = \sum_{\rho=0}^M \sum_{\nu=0}^N a_{\rho\nu} e_\gamma^{(\rho)} d_\mu^{(\nu)}$. The triangle inequality for norms yields, in view of (6), (7), (7.11), and (7.2)

$$(8)\qquad \begin{aligned} &\left\| h_\rho(t) g_\nu(s) - \sum_0^A \sum_0^B e_\gamma^{(\rho)} d_\mu^{(\nu)} \phi_\gamma(t) \psi_\mu(s) \right\| \\ &\leqq \left\| h_\rho(t) \left(g_\nu(s) - \sum_0^B d_\mu^{(\nu)} \psi_\mu(s) \right) \right\| \\ &\qquad + \left\| \sum_0^B d_\mu^{(\nu)} \psi_\mu(s) \left(h_\rho(t) - \sum_0^A e_\gamma^{(\rho)} \phi_\gamma(t) \right) \right\| \\ &\leqq \|h_\rho(t)\|_{I_t} \left\| g_\nu(s) - \sum_0^B d_\mu^{(\nu)} \psi_\mu(s) \right\|_{I_s} \\ &\qquad + \left\| \sum_0^B d_\mu^{(\nu)} \psi_\mu(s) \right\|_{I_s} \left\| h_\rho(t) - \sum_0^A e_\gamma^{(\rho)} \phi_\gamma(t) \right\|_{I_t} \\ &\leqq 2\delta, \qquad \text{for } \rho = 0, 1, \cdots, M, \nu = 0, 1, \cdots, N. \end{aligned}$$

On combining the various inequalities above

$$(9)\qquad \begin{aligned} &\left\| f(s,t) - \sum_0^A \sum_0^B \beta_{\gamma\mu} \phi_\gamma(t) \psi_\mu(s) \right\| \\ &\leqq \left\| f(s,t) - \sum_0^M \sum_0^N a_{\rho\nu} h_\rho(t) g_\nu(s) \right\| \\ &\qquad + \left\| \sum_0^M \sum_0^N a_{\rho\nu} \left(h_\rho(t) g_\nu(s) - \sum_0^A \sum_0^B e_\gamma^\rho d_\mu^\nu \phi_\gamma(t) \psi_\mu(s) \right) \right\| \\ &\leqq \epsilon/2 + \sum_0^M \sum_0^N \left(|a_{\rho\nu}| \left\| h_\rho(t) g_\nu(s) - \sum_0^A \sum_0^B e_\gamma^{(\rho)} d_\mu^{(\nu)} \phi_\gamma(t) \psi_\mu(s) \right\| \right) \\ &\leqq \epsilon/2 + 2\delta \sum_0^M \sum_0^N |a_{\rho\nu}| \leqq \epsilon. \end{aligned}$$

This asserts the closure property for $\{\phi_\gamma(t)\psi_\mu(s)\}$.

The necessity demonstration is equally direct. A trivial application of Fubini's theorem indicates that $\phi_\gamma(t) \in L_2(I_t)$, $\psi_\mu(s) \in L_2(I_s)$ when $\phi_\gamma(t)\psi_\mu(s) \in L_2(I_2)$. No generality is lost if we assume that $\{\psi_\mu(s)\}$ is a linearly independent set of functions. Suppose $\{\psi_\mu(s)\}$ does not have the closure property. Then $f(s) \in L_2(I_s)$ exists for which for all R and b_μ

$$\text{G.L.B.}\left\| f(s) - \sum_0^R b_\mu \psi_\mu(s) \right\|_{I_s} = c > 0, \qquad b_\mu = b_\mu' + i b_\mu''. \tag{10}$$

A fundamental result of Riesz guarantees the existence of *minimal* constants,[6] $\{\bar{b}_\mu^R\}$, such that for $b_\mu \not\equiv b_\mu^R$, $\mu \leqq R$,

$$\left\| f(s) - \sum_0^R \bar{b}_\mu^R \psi_\mu(s) \right\|_{I_s} \leqq \left\| f(s) - \sum_0^R b_\mu \psi_\mu(s) \right\|_{I_s}. \tag{11}$$

The corresponding minimal constants for $Af(s)$ are evidently $\{A\bar{b}_\mu^R\}$. Hence[7]

$$\left\| F(t)f(s) - \sum_0^R \bar{b}_\mu^R F(t)\psi_\mu(s) \right\|_{I_s} \leqq \left\| F(t)f(s) - \sum_0^R b_\mu(t)\psi_\mu(s) \right\|_{I_s}, \quad t \in I_t, \tag{12}$$

when $F(t) \in L_2(I_t)$ is a fixed function of positive norm. We write

$$b_\mu(t) = \sum_0^Q a_{\gamma\mu}\phi_\gamma(t), \qquad Q < \infty. \tag{13}$$

In view of (12) we have

$$\begin{aligned} 0 < c\|F(t)\|_{I_t} &\leqq \left\| f(s)F(t) - \sum_0^R \bar{b}_\mu^R F(t)\psi_\mu(s) \right\| \\ &= \left[\int_{I_t} \left\| f(s)F(t) - \sum_0^R \bar{b}_\mu^R F(t)\psi_\mu(s) \right\|_{I_s}^2 dI_t \right]^{1/2} \\ &\leqq \left[\int_{I_t} \left\| f(s)F(t) - \sum_0^R \sum_0^Q a_{\gamma\mu}\phi_\gamma(t)\psi_\mu(s) \right\|_{I_s}^2 dI_t \right]^{1/2} \\ &= \left\| f(s)F(t) - \sum_0^R \sum_0^Q a_{\gamma\mu}\phi_\gamma(t)\psi_\mu(s) \right\|. \end{aligned} \tag{14}$$

Since (14) is in contradiction with the assumed closure property of $\{\phi_\gamma(t)\psi_\mu(s)\}$ our necessity proof is complete.

We denote by $h_\rho'(t)$ and $g_\nu'(s)$ the step functions in R_n and R_m analogous to $h_\rho(t)$ and $g_\nu(s)$. According to a classical result, $\{h_\rho'(t)g_\nu'(s)\}$, $\rho, \nu = 0, 1, \cdots$, have the closure property in $L_2(E_2)$ when the s, t integration is over R_{n+m} or any Lebesgue measurable subset E_2. Accordingly Theorem 1 and its demonstration remain formally valid in detail when I_s, I_t and I_2 are replaced either by R_n, R_m and R_{n+m} or by

[6] F. Riesz, Acta Mathematica, vol. 41 (1916), p. 77, Lemma 3.

[7] With the choice $F(t)f(s)$, the method of proof of the necessity condition remains valid when sets of infinite measure are included.

the sets E_s, E_t and $E_2 = E_s \times E_t$ of finite or infinite Lebesgue measure.

THEOREM 2. *If* $\{F_\rho(s, t)\}$, $\rho = 0, 1, \cdots$, *is closed in* $L_2(E_2)$, *then the sequence is also closed in* $L_2(E_s)$ *except possibly for a t set of zero measure.*[8]

Suppose a lower bound of approximation to $f(s) \varepsilon L_2(E_s)$, by linear combinations of $\{F_\rho(s, t)\}$, is $c(t) \varepsilon L_2(E_t)$, where $\infty > c(t) > 0$ for $t \varepsilon G \subset E_t$. Let $F(t) \varepsilon L_2(E_t)$ differ from 0 on G (say $F(t) = c(t)$). The analogue of (14) is

$$\|c(t)F(t)\|_G \leqq \|c(t)F(t)\|_{E_t} \leqq \left\| f(s)F(t) - \sum_0^R b_\mu F_\mu(s, t) \right\|. \tag{14'}$$

Hence G has zero measure. Let $\{f^{(\sigma)}(s)\}$ be closed in $L_2(E_s)$ and denote the corresponding G sets, defined above, by $\{G^\sigma\}$. The denumerable sum $\mathfrak{S}G^\sigma$ is plainly of measure zero. Thus $\{F_\rho(s, t)\}$ is closed in $L_2(E_s)$ for all $t \varepsilon E_t - \mathfrak{S}G^\sigma$.

We now abstract the properties needed in the foregoing proofs. Let $T(E)$ denote a Banach space[9] of real functions on E. A set G, $G \subset E$, will be called a *non-significant* set if $f(z) \varepsilon T(E)$ may be arbitrarily changed on G without affecting the value of $\|f(z)\|_E$. The postulates below hold for $T(E)$. When (d) and (e) are omitted we write $T_-(E)$.

(a) If $f(s, t) \varepsilon T(E_2)$ then $f(s, t) \varepsilon T(E_s)$ and $f(s, t) \varepsilon T(E_t)$ for all save a non-significant set of t or s values respectively. If $f(s) \varepsilon T(E_s)$, $F(t) \varepsilon T(E_t)$ then $f(s)F(t) \varepsilon T(E_2)$.

(b) $\|f(s, t)\|_{E_2} = \| \|f(s, t)\|_{E_s}\|_{E_t}$.

(c) If, neglecting non-significant sets, $|f_1(t)| > |f_2(t)|$, then $\|f_1(t)\|_{E_t} > \|f_2(t)\|_{E_t}$.

(d) There exists a sequence $\{h_\rho(t)g_\nu(s)\}$, $\rho, \nu = 0, 1, \cdots$, with the closure property in $T(E_2)$, where $h_\rho(t) \varepsilon T(E_t)$ and $g_\nu(s) \varepsilon T(E_s)$.

(e) Denumerable sums of non-significant sets are non-significant sets.

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[8] A sharper result follows from Fatou's lemma. Suppose $F(t) \varepsilon L_2(E_t)$ differs from 0 for almost all $t \varepsilon E_t$. Now

$$0 = \mathrm{L}_{N\to\infty}\left\|f(s)F(t) - \sum_0^N b_\rho^{(N)} F_\rho(s, t)\right\|_{E_2}^2 \geqq \int_{E_t} \underline{\mathrm{L}}_{N\to\infty}\left\|f(s)F(t) - \sum^N b_\rho^{(N)} F_\rho(s, t)\right\|_{E_s}^2 dE_t.$$

Thus a suitable sequence $\{\sum^{N_i} b_\rho^{(i)} F_\rho(s, t)\}$, with *constant coefficients* $\{b_\rho^{(i)}\}$, converges strongly to $f(s)$ in $L_2(E_s)$ for almost all $t \varepsilon E_t$. Moreover if E_t is of finite measure, the Egoroff theorem guarantees uniform convergence for $t \varepsilon D_\delta \subset E_t$ where the measure of $E_t - D_\delta$ is inferior to arbitrary δ. A closed sequence $\{f_\sigma(s)\}$ is introduced as above.

[9] S. Banach, *Théorie des Opérations Linéaires*, pp. 53, 58. Banach uses *fundamental* in the sense of our *closed.*

THEOREM 3. (α) *If* $\{\phi_\gamma(t)\}$ *and* $\{\psi_\mu(s)\}$ *are closed in* $T(E_t)$, $T(E_s)$ *then* $\{\phi_\gamma(t)\psi_\mu(s)\}$ *is closed in* $T(E_2)$. (β) *If* $\{\phi_\gamma(t)\psi_\mu(s)\}$ *is closed in* $T_-(E_2)$, *then* $\{\psi_\mu(s)\}$ *is closed in* $T_-(E_s)$. (γ) *If* $\{F_\rho(s, t)\}$ *is closed in* $T(E_2)$, *then* $\{F_\rho(s, t)\}$ *is closed in* $T(E_s)$ *for all but a non-significant set of t values in* E_t.

The demonstrations of Theorems 1 and 2 apply without change in form.[10] The space[11] $L_p(E, \mu)$, $p \geqq 1$, is included in $T(E)$. This is the space of measurable functions whose pth powers are summable over the measurable set E, where the Lebesgue-Radon-Stieltjes integral is equally admissible with the usual Lebesgue integral. Thus the symbol $\mu(E)$ denotes either the Lebesgue measure, or the Radon measure determined by a non-negative additive function of intervals. In all cases $\mu_2(E_2) = \mu_s(E_s)\mu_t(E_t)$, and the sets of zero measure constitute the non-significant sets. The norm is

$$\|f(s, t)\| = \left[\int_E |f|^p d\mu(E) \right]^{1/p}. \tag{15}$$

The verification of the main postulates is implied by the Fubini theorem, the Hölder-Minkowski inequalities and the denseness of the step functions. The functions $\{h_\rho(t)\}$, $\{g_\nu(s)\}$ or $\{h_\rho'(t)g_\nu'(s)\}$ as defined in Theorem 1 are again available.[12]

The space $C(E)$ of continuous functions is another special case of $T(E)$. We assume $E_s \subsetneqq R_m$, $E_t \subsetneqq R_n$ and $E_2 \subsetneqq R_{n+m}$ are bounded closed sets. The null set is the only non-significant set. The norm is

$$\|f(s, t)\| = \max_{s, t \in E_2} |f(s, t)|. \tag{16}$$

The sequences $h_\rho(t)$ and $g_\nu(s)$ are the ordered products of the elements $1, t_1, \cdots, t_n$ and of $1, s_1, \cdots, s_m$ respectively.

Postulates (b) and (c) may be replaced by the weaker

(b′) $\|f(w, z)\|_{E_2} < \epsilon$ implies $\|f(w, z)\|_{E_w} < \eta(\epsilon)$, where $L_{\epsilon \to 0}\eta(\epsilon) = 0$ except possibly for non-significant z sets.

(c′) $\|G(z)\|_{E_z} = 1$, $\|H(w)\|_{E_w} < \epsilon$ imply $\|G(z)H(w)\|_{E_2} < \eta(\epsilon)$.

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[10] For (α), postulate (d) may be replaced by the assumption that each $f(s, t) \in T(E_2)$ is the strong limit of some (not necessarily fixed) sequence $\{h_\rho^f(t)g_\nu^f(s)\}$, where $h_\rho^f(t) \in T(E_t)$ and $g_\nu^f(s) \in T(E_s)$.

[11] Saks, loc. cit. (1928), chap. 3, or J. Radon, Sitzungsberichte der Akademie der Wissenschaften, Vienna, class IIa, vol. 122 (1913). The Lebesgue case admits sets of infinite measure.

[12] For $p > 1$ a valid theorem on *completeness* is obtained from Theorem 3 if closure (in $L_p(E, \mu)$) is replaced by completeness in $L_{p/(p-1)}(E, \mu)$ where E refers to E_s, E_t and E_2 in turn.

These modifications will be connoted by writing $T'(E)$ and $T'_-(E)$. Consider, for instance, $C^1(E)$, the space of functions continuous together with their first partial derivatives on[13] E. We restrict ourselves now to *closed* linear intervals I_s, I_t and the rectangle I_2: $I_s \times I_t$. The norms in $C^1(I_2)$ and $C^1(I_s)$ are,[14] with $f_s \equiv \partial f/\partial s$,

$$(17)\qquad \begin{aligned} \|f(s,t)\| &= \max_{I_2} |f(s,t)| + \max_{I_2} |f_s(s,t)| + \max |f_t(s,t)|, \\ \|f(s)\| &= \max_{I_s} |f(s)| + \max_{I_s} |f_s(s)|. \end{aligned}$$

It is well known that $C^1(I_s)$ (and $C^1(I_t)$) is complete. It is easy to show that $C^1(I_2)$ also is complete. Indeed if $\{f^{(n)}(s,t)\}$ is a Cauchy sequence in $C^1(I_2)$, then $f^{(n)}(s,t)$, $f_s^{(n)}(s,t)$ and $f_t^{(n)}(s,t)$ converge *uniformly* in I_2 and hence define an element of $C^1(I_2)$.

Since

(b′) $\|F(s,t)\|_{I_2} \geqq \max_{t \in I_t} \|f(s,t)\|_{I_s}$ (t and s are interchangeable),

(c′) $\|G(s)H(t)\|_{I_2} \leqq \|G(s)\|_{I_s} \|H(t)\|_{I_t}$.

it is clear that (b′) and (c′) are satisfied.

THEOREM 4. *The conclusions in* (α), (β), (γ) *of Theorem* 3 *remain valid when* $T'(E)$ *and* $T'_-(E)$ *replace* $T(E)$ *and* $T_-(E)$.

For (α) we now choose δ small enough in (7) and (7.1) to yield $\eta(\delta)$ inferior to the right side of (6). Then (6.1) is valid with $\eta(\delta)$ written in place of δ. On making use of (c′) it is easily shown that the left side of (8) is smaller than $2\eta(\delta)$ and the final inequality in (9) is again obtained. For (β) we need only change (14) slightly. Indeed, by reference to (b′) and (13)

$$\epsilon \geqq \left\| f(s)F(t) - \sum\sum a_{\gamma\mu}\phi_\gamma(t)\psi_\mu(s) \right\|_{I_2}$$

would imply the contradiction

$$(14'')\qquad \eta(\epsilon) \geqq |c| \text{ true max } |F(t)| > 0.$$

The *true maximum* is defined just as in the analogous case of measurable functions and implies neglect of non-significant t sets. Evidently (γ) also may be maintained. Indeed the argument in footnote 8, for ex ple, is easily amended to yield the desired result.

If the closure property of the sequence $\{\phi_\rho(z)\}$ in $T_-(E)$ or $C^1(I)$ is unaffected by the omission of $\phi_\sigma(z)$, then we shall say $\{\phi_\rho(z)\}$ is a

[13] The sets used in $C(E)$ are available for $C^1(E)$ also.

[14] Even if $f(s,t)$ and $g(s)h(t) \in C^1(E_2)$, $\| \|f(s,t)\|_{I_s}\|_{I_t}$ and $\| \|g(s)h(t)\|_{I_s}\|_{I_t}$ need not exist. Thus $C^1(E)$ is not included under $T(E)$.

"*redundant*" sequence and $\phi_\sigma(z)$ is a "*superfluous*" function. If $\{\phi_{\sigma_k}\}$, $k=1, 2, \cdots, K$, is superfluous, then for arbitrary ϵ we can satisfy

$$\left\| \phi_{\sigma_k} - \sum_0^N{}' c_j\phi_j(z) \right\| < \epsilon, \quad j \neq \sigma_l, l = 1, \cdots, K. \tag{18}$$

LEMMA 1. *In $T_-(E)$ or $T'_-(E)$ if $\{f_\mu(z)\}$ is closed and non-redundant, then for any $F(z)$, $\overline{\mathrm{L}}_{\epsilon\to 0}|d_1(\epsilon)| \leqq D < \infty$ where $d_1(\epsilon)$ is consistent with $\|F(z) - d_1(\epsilon)f_1(z) - \sum_2^N d_j f_j(z)\| < \epsilon$.*

In the contrary case

$$\begin{aligned} \epsilon + \|F(z)\| &\geqq \|F(z)\| + \left\| F(z) - d_1(\epsilon)f_1(z) - \sum_2^N d_j f_j(z) \right\| \\ &\geqq |d_1(\epsilon)| \left\| f_1(z) - \sum_2^N \frac{d_i}{d_1} f_i(z) \right\|. \end{aligned} \tag{19}$$

Now

$$\left\| f_1(z) - \sum_2^N \frac{d_i}{d_1} f_i(z) \right\| \geqq c > 0, \tag{20}$$

for all N and d_i, since $f_1(z)$ is not superfluous. For all sufficiently small ϵ, (19) and (20) imply

$$|d_1(\epsilon)| \leqq 2\|F(z)\|/c \tag{21}$$

in contradiction with the hypothesized non-boundedness of $d_1(\epsilon)$.

THEOREM 5. *If $\{\phi_\mu(t)\psi_\mu(s)\}$ is closed in $T_-(E_2)$ or C^1E, then* (I) *$\{\psi_\mu(s)\}$ is closed in $T_-(E_s)$ (or $C^1(I_s)$);* (II) *every finite subsequence of $\{\psi_\mu(s)\}$ is superfluous.*[15]

Evidently (I) is a special case of Theorem 3(β). In view of (I) if $\phi_\sigma(T)\psi_\sigma(s)$, $\sigma=1, \cdots, q$, is superfluous, then $\psi_\sigma(s)$, $\sigma=1, \cdots, q$, is superfluous. Accordingly we may restrict ourselves to non-redundant sequences $\{\phi_\mu(t)\psi_\mu(s)\}$.

We demonstrate (II) by induction. Suppose $\psi_1(s), \cdots, \psi_{n-1}(s)$ are superfluous. Since no finite basis exists in $T_-(E)$ or $C^1(I)$, we may find a function $F(t)$ such that the set $F(t), \phi_\sigma(t)$, $\sigma=1, \cdots, n$, is linearly independent. Suppose $\psi_n(s)$ is not superfluous. Then

$$\left\| \psi_n(s) - \sum_{n+1}^N k_i\psi_i(s) \right\|_E \geqq c > 0, \tag{22}$$

[15] Evidently $\{\psi_\mu(s)\}$ need not be *dense closed* in the sense that any infinite subsequence is closed.

for all k_i and N. By hypothesis sequences $\{a_i^{(\rho)}\}$ and a constant N exist for arbitrary ϵ such that

$$\left\| \psi_\rho(s) - \sum_{i=n}^{N} a_i^{(\rho)} \psi_i(s) \right\|_{E_s} \leqq \epsilon, \quad \rho = 1, \cdots, n-1. \tag{22.1}$$

Moreover

$$\begin{aligned} &\left\| \psi_n(s)F(t) - \sum_{\rho=1}^{n} d_\rho \phi_\rho(t)\psi_\rho(s) - \sum_{n+1}^{N} d_i\phi_i(t)\psi_i(s) \right\| \\ &\quad + \left\| \sum_{\rho=1}^{n-1} d_\rho \phi_\rho(t) \left(\psi_\rho(s) - \sum_{n}^{N} a_i^{(\rho)} \psi_i(s) \right) \right\| \\ &\geqq \left\| \psi_n(s) \left[F(t) - \sum_{\rho=1}^{n-1} d_\rho a_n^{\rho} \phi_\rho(t) - d_n\phi_n(t) \right] \right. \\ &\quad \left. - \sum_{n+1}^{N} d_i\phi_i(t)\psi_i(s) - \sum_{\sigma=1}^{n-1} \sum_{n+1}^{N} a_i^{(\sigma)} d_\sigma \phi_\sigma(t)\psi_i(s) \right\|. \end{aligned} \tag{23}$$

The right side of this inequality, by an argument similar in all details to that involved in the passage from (12) to (14), dominates

$$c \left\| F(t) - \sum_{1}^{n-1} d_\sigma a_n^{(\sigma)} \phi_\sigma(t) - d_n\phi_n(t) \right\|_{I_t} \text{ in } T_{-}(E_2) \tag{23.1}$$

or (cf. (b″))

$$\max_{t \,\varepsilon\, I_t} c \left| F(t) - \sum_{1}^{n-1} d_\sigma a_n^{\sigma} \phi_\sigma(t) - d_n\phi_n(t) \right| \text{ in } C^1(I_2). \tag{23.2}$$

In (23.2) we note $\phi_j(t) \,\varepsilon\, C^1(I_t)$ implies $\phi_j(t) \,\varepsilon\, C(I_t)$. Hence again by the Riesz theorem the expressions in (23.1) and (23.2) have a positive lower bound, denoted by $K>0$. In view of (22.1) closure of $\{\phi_\mu(t)\psi_\mu(s)\}$ and postulates (b) or (c″), constants N, d_i and $a_i^{(\rho)}$ exist such that the left side of (23) is inferior to

$$\epsilon + \sum_{1}^{n-1} | d_\rho | \, \|\phi_\rho(t)\|_{I_t} \epsilon. \tag{24}$$

Hence by Lemma 1 applied to each d_ρ the upper bound in (24) approaches 0 with ϵ in contradiction with the conclusion $K>0$. Thus $\psi_n(s)$ is superfluous.

This type of argument may be used to show that the non-redundancy of $\{\phi_\mu(t)\psi_\mu(s)\}$ implies that $\psi_1(s)$ is superfluous. The induction is thus complete and part (II) of our theorem is established. It is an easy matter to extend the theorem to $T'_{-}(E)$ spaces.

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