THE SPACES L^p WITH $0<p<1$[1]

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If we have a Lebesgue measurable set E in any n-dimensional euclidean space, and have any positive number q, we define $L^q(E)$ to be the class of all real-valued Lebesgue measurable functions f on E for which $\int_E|f|^q<\infty$. As is well known, whenever $q\geqq 1$ the class of functions $L^q(E)$ is a Banach space with the norm $||f||_q=(\int_E|f|^q)^{1/q}$. When $0<p<1$, the function $||f||_p$ no longer satisfies the triangle inequality $||f_1+f_2||_p\leqq||f_1||_p+||f_2||_p$ but in general only the weaker condition $||f_1+f_2||_p\leqq 2^\nu[||f_1||_p+||f_2||_p]$, where $\nu=(1-p)/p$. (This can be shown by considering the function $(1+x^p)/(1+x)^p$.) If we consider such an L^p space as a linear topological space in which the neighborhoods of a point f_0 are the spheres of radius $\epsilon>0$, $E_{f\,\varepsilon L^p}[||f-f_0||_p<\epsilon]$, it follows from theorems of Hyers and Wehausen[3] that this topology can be given by an equivalent Fréchet metric. This suggests that while many theorems on Banach spaces which can be applied to the spaces $L^p(E)$ with $p\geqq 1$ may fail to hold in those spaces with $0<p<1$, there may still remain many theorems on Fréchet spaces and pseudonormed spaces which may be applicable. However, Theorem 1 shows that almost no results depending on the use of linear (that is, additive and continuous) functionals can be usefully applied in these spaces.

THEOREM 1. *Any linear functional on $L^p(E)$ is identically zero.*

The proof of this and of some additional results is given in a series of lemmas using a more general set of assumptions. We assume as a background some knowledge of the first chapter of Saks[4] book, in which he deals with what he calls "the integral," a completely additive integral having many of the properties of the Lebesgue integral. We consider a set Y of elements y, an additive family[5] $\mathfrak{X}$ of subsets of Y and an additive, non-negative[6] set-function μ on $\mathfrak{X}$ such that $\mu(Y)<\infty$. This last condition will be imposed from here until we

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[3] D. H. Hyers, *A note on linear topological spaces*, this Bulletin, vol. 44 (1938), pp. 76–80, and J. V. Wehausen, *Transformations in linear topological spaces*, Duke Mathematical Journal, vol. 4 (1938), pp. 157–169.

[4] S. Saks, *Theory of the Integral*, Warsaw, 1937.

[5] A class $\mathfrak{X}$ of subsets of Y is called an additive family if (a) $X\,\varepsilon\,\mathfrak{X}$ implies $Y-X\,\varepsilon\,\mathfrak{X}$, and (b) $X_n\,\varepsilon\,\mathfrak{X}$ implies $\sum_{n<\infty}X_n\,\varepsilon\,\mathfrak{X}$.

[6] A set-function μ is additive if whenever X_n are disjoint sets of $\mathfrak{X}$ then $\mu(\sum_{n<\infty}X_n)=\sum_{n<\infty}\mu(X_n)$; μ is non-negative if $\mu(X)\geqq 0$ for every $X\,\varepsilon\,\mathfrak{X}$.

reach Theorem 4. We shall finally remove it altogether and get the same type of theorem giving the linear functionals on any $L^p_\mu(Y)$. We shall also find the conjugate space of the sequence spaces related to these. For each $p>0$ let $L^p = L^p_\mu(Y)$ be the class of real-valued, $\mathfrak{X}$-measurable[7] functions f on Y such that $\|f\|_p = (\int_Y |f|^p d\mu)^{1/p} < \infty$. It is clear, as in the spaces with $p \geqq 1$, that if U is a linear functional on L^p, there is a $K \geqq 0$ such that $|U(f)| \leqq K$ whenever $\|f\|_p \leqq 1$. We let $\|U\|$ be the smallest such K; that is, $\|U\| = \sup_f |U(f)|$ where the sup is taken over those f with $\|f\|_p = 1$. Similarly if an additive functional is bounded on the unit sphere it is easily seen to be continuous and hence linear.

In the series of lemmas immediately following we shall consider a fixed number p with $0<p<1$ and a fixed family $\mathfrak{X}$ and measure μ.

LEMMA 1. *If $f_i, f \in L^1$ and $\|f_i - f\|_1 \to 0$, then $\|f_i - f\|_p \to 0$.*

PROOF. Let $E_i = E_{y \in Y}[|f_i(y) - f(y)| > 1]$, and let $g_i(y) = f(y)$ in E_i, $g_i(y) = f_i(y)$ in $Y - E_i$. Then

$$\int_Y |f_i - f|^p d\mu = \int_{E_i} |f_i - f|^p d\mu + \int_Y |g_i - f|^p d\mu.$$

We have $\int_{E_i} |f_i - f|^p \leqq \int_{E_i} |f_i - f| d\mu \leqq \int_Y |f_i - f| d\mu$; this integral tends to zero by hypothesis. Also $\|g_i - f\|_1 \to 0$, so g_i converges in measure to f; moreover, $|g_i(y) - f(y)| \leqq 1$ for all $y \in Y$, so, by the Lebesgue convergence theorem, $\int_Y |g_i - f|^p d\mu \to 0$.

LEMMA 2. *Every linear functional on L^1 can be expressed in the form $U(f) = \int_Y fu d\mu$ where u depends on U, is $\mathfrak{X}$-measurable and, except possibly on a set of μ-measure zero, is bounded.*

The proof follows that given in Banach[8] for the special case $Y = (0, 1)$. If we let f_E be the characteristic function of $E \in \mathfrak{X}$ and let $\Phi(E) = U(f_E)$, the set-function Φ thus defined on $\mathfrak{X}$ is additive, $|\Phi(E)| \leqq \|U\| \mu(E)$ and $U(f) = \int_Y f d\Phi$ for each $f \in L^1$. By the theorem of Radon and Nikodym (Saks, p. 36) there is an $\mathfrak{X}$-measurable function u such that $\Phi(E) = \int_E u d\mu$ and $\|U\| = \text{ess sup}_{y \in Y} |u(y)|$. By a theorem on change of measure (Saks, p. 37) we have $U(f) = \int_Y fu d\mu$.

LEMMA 3. *If U is linear on L^p, there is an $\mathfrak{X}$-measurable, essentially bounded function u on Y such that whenever $f \in L^1$, $U(f) = \int_Y fu d\mu$, while if $f \in L^p - L^1$ there exists a sequence $\{f_i\}$ of functions in L^1 such that*

[7] A function f defined on Y is $\mathfrak{X}$-measurable if for each real a the set $E_{y \in Y}[f(y) > a]$ is in $\mathfrak{X}$.

[8] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932.

$\|f_i - f\|_p \to 0$, *and for every such sequence* $U(f) = \lim_{i\to\infty} U(f_i) = \lim_{i\to\infty} \int_Y f_i u d\mu$.

This follows immediately from the first two lemmas and from the obvious fact that the elements of L^1 form a dense set in L^p.

LEMMA 4. *If for each* $E \varepsilon \mathfrak{X}$ *with* $\mu(E) > 0$ *there is a function* $f_0 \varepsilon L^p - L^1$ *such that* $f_0(y) = 0$ *when* $y \varepsilon Y - E$, *and if* U *is linear on* L^p, *then* $U(f) = 0$ *for every* $f \varepsilon L^p$.

PROOF. If U is not identically zero, the function u of Lemma 3 cannot be almost everywhere zero; hence there is an $\alpha > 0$ and a set $E \varepsilon X$ such that $|u(y)| > \alpha$ if $y \varepsilon E$. Let $f'(y) = f_0(y)$ signum $f_0(y) \cdot u(y)$ where signum $s = |s|/s$ when $s \neq 0, = 0$ when $s = 0$; for each n let $E_n = E_{y \varepsilon Y}[|f'(y)| \leqq n]$, and let $f_n(y) = f'(y)$ on E_n, $f_n(y) = 0$ otherwise. Then $\|f_n - f'\|_p \to 0$; also

$$U(f_n) = \int_Y u f_n d\mu = \int_{E_n} |f_0 u| \, d\mu \geqq \alpha \int_{E_n} |f_0| \, d\mu,$$

but $E_n \uparrow E$ and $\int_E |f_0| d\mu = \infty$; so $U(f_n) \uparrow \infty$, contradicting continuity.

LEMMA 5. *If* $q > 1$, $E \varepsilon \mathfrak{X}$ *and* $\mu(E) > 0$, *and if no function* f *which is zero on* $Y - E$ *belongs to* $L^1 - L^q$, *then every* $f \varepsilon L^1$ *is essentially bounded on* E.

PROOF. If there is a function not essentially bounded on E, there is such a function f_0 which at the same time is zero on $Y - E$. Hence there exists a countably infinite sequence of sets $E_n \varepsilon \mathfrak{X}$ with $\mu(E_n) = b_n > 0$, $E_n \subset E$, E_n disjoint, $|f_0(y)| \geqq n$ on E_n, and, since $f_0 \varepsilon L^q$, $\sum n^q b_n < \infty$. Let α be so chosen that $1 < \alpha < 1/q + q - 1$ and let $a_n = 1/b_n n^\alpha$; let $f(y) = a_n$ if $y \varepsilon E_n$, $f(y) = 0$ if $y \varepsilon Y - \sum_{n<\infty} E_n$. Then $\int_Y |f| d\mu = \sum_{n<\infty} a_n b_n = \sum n^{-\alpha} < \infty$, so $f \varepsilon L^1$. On the other hand $\int_Y |f|^q d\mu = \sum a_n^q b_n = \sum b_n^{1-q} n^{-\alpha q}$. Since $\sum n^q b_n < \infty$ there must be a $K > 0$ such that $n^q b_n < K$ for all n; this implies that $b_n^{1-q} > n^{q(q-1)} K^{1-q}$ so $\int_Y |f|^q d\mu \geqq K^{1-q} \sum n^{q(q-1)} n^{-\alpha q} \geqq K^{1-q} \sum n^{-1} = \infty$, and $f \notin L^q$ contrary to hypothesis.

For convenience we introduce a property of sets of $\mathfrak{X}$: a set $E \varepsilon \mathfrak{X}$ has property A relative to μ (in symbols $E \varepsilon \mathfrak{A}$) if $0 < \mu(E) < \infty$ and for every $E' \varepsilon \mathfrak{X}$ either[9] $\mu(E \cdot E') = 0$ or $\mu(E - E') = 0$.

There are several obvious properties of the sets of $\mathfrak{A}$.

(1) If E has only one element and $\mu(E) > 0$, then $E \varepsilon \mathfrak{A}$.

[9] The referee has called to the writer's attention the fact that S. Saks calls such sets "singular" in his paper, *Addition to the note on some functionals*, Transactions of this Society, vol. 35 (1933), pp. 965–970.

(2) If $E \varepsilon \mathfrak{A}$ and f is $\mathfrak{X}$-measurable, then f is equal to a constant $\bar{f}$ almost everywhere on E.

(3) If $E_1 \varepsilon \mathfrak{A}$ and $E_2 \varepsilon \mathfrak{X}$, either $\mu(E_1E_2)=0$ or $\mu(E_1E_2)=\mu(E_1)$.

(4) $\mathfrak{A}$ is empty if and only if given $\epsilon>0$ it is possible to cover any set $E \varepsilon \mathfrak{X}$ with a countable collection of sets $S_i \varepsilon \mathfrak{X}$ with $\mu(S_i)<\epsilon$ for all i.

(5) If $E \varepsilon \mathfrak{A}$ and $\mu(E')=\mu(E'')=0$, then $E_0=E+E'-E'' \varepsilon \mathfrak{A}$ also.

THEOREM 2. *There is a nonzero linear functional on L^p if and only if $\mathfrak{A}$ is not empty.*

PROOF. If $\mathfrak{A}$ is empty and $E \varepsilon \mathfrak{X}$ with $\mu(E)>0$, there is a sequence of disjoint subsets of E, each of positive measure. Hence there is a function in L^1 essentially unbounded on E; therefore, by Lemma 5 with $q=1/p$, there is an $f \varepsilon L^p-L^1$ with $f(y)=0$ in $Y-E$; by Lemma 4 every linear functional on L^p is identically zero.

On the other hand if there is a set $E \varepsilon \mathfrak{A}$, and if $\|f_n-f\|_p \to 0$, f_n converges in measure to f; that is, for each $\epsilon>0$, $\mu(E_{y\varepsilon Y}[|f_n(y)-f(y)|>\epsilon]) \to 0$ as $n\to\infty$. If $\bar{f}_n$ and $\bar{f}$ are the values of f_n and f almost everywhere on E, it follows that $\bar{f}_n \to \bar{f}$. Hence for any real number k setting $U(f)=k\bar{f}$ defines a linear functional U on L^p and if $k\neq 0$ then U is not identically zero.

For the case in which $m(E)$, the Lebesgue measure of E, is finite, Theorem 1 follows directly from Theorem 2 since every Lebesgue measurable set can be split into two subsets, each of measure half as great, so $\mathfrak{A}$ is empty for Lebesgue measure; the same conclusion could be derived from (4) above. The general case follows from this and from Theorem 6 below.

There are a large number of measures μ for which $\mathfrak{A}$ is empty. For example, to get one such class of measures, let X be a separable metric space and μ a Carathéodory measure on the class $\mathfrak{X}$ of μ-measurable sets; the class $\mathfrak{X}$, as is well known, contains all the Borel sets in X. If Y is a measurable subset of X, we say that μ is a uniformly continuous function on Y if the conditions $E_n \varepsilon \mathfrak{X}$, $E_n \subset Y$, and the sequence of diameters[10] $\delta(E_n)\to 0$ together imply that $\mu(E_n)\to 0$. It is easy to show from this condition that given $\epsilon>0$ there is a $\delta>0$ such that $\mu(E_n)<\epsilon$ whenever $E \varepsilon \mathfrak{X}$, $E \subset Y$ and $\delta(E)<\epsilon$.

LEMMA 6. *If μ is a Carathéodory measure on the separable space X, if $X=\sum Y_j$ with each $Y_j \varepsilon \mathfrak{X}$ and $\mu(Y_j)<\infty$, and μ is uniformly continuous on each Y_j, then the class $\mathfrak{A}$ is empty.*

[10] $\delta(E)=\sup_{x,y}\rho(x,y)$ where the sup is taken over $x \varepsilon E$ and $y \varepsilon E$ and $\rho(x,y)$ is the distance in X from x to y.

PROOF. If $E \varepsilon \mathfrak{A}$ and $\mu(E)=b>0$, we have by (3) that $\mu(EY_i)=0$ if $i\neq j$, while $\mu(EY_j)=b$; then by the separability of X we can cover Y_j with spheres of arbitrarily small radius. Hence by uniform continuity and (4) $\mu(EY_j)=0$ also and $\mu(E)=0$, so E cannot be in $\mathfrak{A}$.

COROLLARY. *For such X and μ, if $\mu(X)<\infty$ there are no nonzero linear functionals on $L^p_\mu(X)$, $0<p<1$.*

We remark that a later theorem allows us to remove the restriction that $\mu(X)<\infty$.

The interesting spaces $L^p_\mu(Y)$, then, are those for which the set $\mathfrak{A}$ is not empty; in such a case the structure of $\mathfrak{A}$ is rather simple.

LEMMA 7. *If $\mathfrak{A}$ is not empty, there exists a finite or countable sequence of sets $\{E_i\}\subset\mathfrak{A}$ such that every $E \varepsilon \mathfrak{A}$ differs from just one E_i only by sets of measure zero.*

PROOF. Well-order $\mathfrak{A}$, and define E_α to be the first set in the ordering which does not overlap any E_β, $\beta<\alpha$, on a set of positive measure. Since $\mu(Y)<\infty$, the number of these sets E_α for which $\mu(E)>1/n$ must be finite for each n; hence the sequence $\{E_\alpha\}$ is finite or countable.

In what follows we shall let $\{E_i\}$ be the family of sets of $\mathfrak{A}$ of the preceding lemma, let $b_i=\mu(E_i)$ and let $\bar{f}_i$ be the value of the measurable function f almost everywhere on E_i.

THEOREM 3. *Any linear functional on L^p is identically zero, if $\mathfrak{A}$ is empty, or can be expressed in the form $U(f)=\sum_i u_i\bar{f}_i$ where $\|U\|=\sup_i |u_i| b_i^{-1/p}$. If u_i are given so that $|u_i|<Kb_i^{1/p}$ for all i, the functional $U(f)=\sum_i u_i\bar{f}_i$ is linear on L^p.*

PROOF. It is easily seen from Theorem 2, applied to $L^p_\mu(Y-\sum_i E_i)$, that $U(f)$ is independent of the values of f on $Y-\sum_i E_i$; hence no generality is lost if we assume $Y=\sum_i E_i$ for simplicity. Let $f_{(i)}$ be the characteristic function of E_i (that is, $f_{(i)}(y)=1$ if $y \varepsilon E_i$, $f_{(i)}(y)=0$ otherwise), and take any $f \varepsilon L^p$; then $\|\sum_{i\leq n}\bar{f}_i f_{(i)}-f\|_p\to 0$ as $n\to\infty$; so, by continuity, $U(f)=\lim_n U(\sum_{i\leq n}\bar{f}_i f_{(i)})=\lim_n\sum_{i\leq n}\bar{f}_i U(f_{(i)})=\sum_i u_i\bar{f}_i$ where $u_i=U(f_{(i)})$. Postponing the computation of $\|U\|$ for a moment, let us assume u_i given such that $|u_i|\leq Kb_i^{1/p}$, and take $\|f\|_p=1$. Then $|\bar{f}_i b_i^{1/p}|\leq\|f\|_p\leq 1$, so that $|\bar{f}_i| b_i^{1/p}\leq|\bar{f}_i|^p b_i$ since $0<p<1$. Therefore $|U(f)|\leq\sum_i|u_i\bar{f}_i|\leq K\sum_i|b_i^{1/p}\bar{f}_i|\leq K\sum_i|\bar{f}_i|^p b_i\leq K$, so U is bounded; hence it is continuous, and $\|U\|\leq K$. It follows, if $K=\sup_i |u_i| b_i^{-1/p}$, that $\|U\|\geq K$ also for $|U(b_i^{-1/p}f_{(i)})|=|u_i| b_i^{-1/p}\leq\|U\|$; therefore $\sup_i|u_i|b_i^{-1/p}\leq\|U\|$ also. We note that this shows that the series $\sum_i|u_i\bar{f}_i|$ converges.

We now turn to the case in which $\mu(Y)$ need not be finite; for greater convenience we introduce another property of sets relative to μ; Y has property B (in symbols $Y \varepsilon \mathfrak{B}$) if Y can be expressed as $\sum_{j<\infty} Y_j$ with $Y_j \varepsilon \mathfrak{X}$ and $\mu(Y_j) < \infty$ for each j.

Clearly Lemma 7 holds if $Y \varepsilon \mathfrak{B}$ as well as if $\mu(Y) < \infty$; we let E_i, b_i and $\bar{f}_i$ have their previous meanings.

THEOREM 4. *If $Y \varepsilon \mathfrak{B}$, a functional U on $L^p_\mu(Y)$ is linear if and only if* (a) *it is identically zero and $\mathfrak{A}$ is empty, or* (b) *$\mathfrak{A}$ is not empty and U can be expressed in the form $U(f) = \sum_i u_i \bar{f}_i$ with $\|U\| = \sup_i |u_i| b_i^{-1/p}$.*

PROOF. The sets Y_j which exist by property B can obviously be taken disjoint; we let U_j be a linear functional on $L^p_\mu(Y)$ defined by $U_j(f) = U(f_j')$ where $f_j'(y) = f(y)$ when $y \varepsilon Y_j$, $f_j'(y) = 0$ otherwise. Then U_j is linear on $L^p_\mu(Y_j)$; so by Theorems 2 and 3 either U_j is identically zero or $U_j(f) = \sum_i u_{ji} \bar{f}_{ji}$. Now $\lim_n \|\sum_{i \leqq n} f_i' - f\|_p = 0$, so $U(f) = \lim_n \sum_{j \leqq n} U(f_j') = \sum_j U_j(f)$. Moreover there is an $f_0 \varepsilon L^p$ such that $|f_0(y)| = |f(y)|$ for all $y \varepsilon Y$ and $U_j(f_0) = |U_j(f)|$ for all j; hence $|U_j(f)| \leqq \sum_j U_j(f_0) = U(f_0) \leqq \|U\| \cdot \|f_0\| = \|U\| \cdot \|f\|$, so $\|U_j\| \leqq \|U\|$ and the series $\sum_j U_j(f)$ converges absolutely for all $f \varepsilon L^p_\mu$. Then $|u_{ji}| b_{ji}^{-1/p} \leqq \|U_j\| \leqq \|U\|$ and $U(f) = \sum_j \sum_i u_{ji} \bar{f}_{ji}$ unless all U_j are identically zero, that is, unless $\mathfrak{A} = 0$. Rearranging this into a simple series, permissible since it converges absolutely, we get $U(f) = \sum u_i f_i$. The other conclusions follow as in Theorem 3.

The general form of Theorem 1 follows directly from this, since the whole euclidean space is the sum of a countable set of finite intervals.

There remains the case in which Y is not in $\mathfrak{B}$. As we did in the proof of Lemma 7, we can define a well-ordered set $\{E_\gamma\}$ of sets of $\mathfrak{A}$, disjoint up to sets of measure zero and such that every $E \varepsilon \mathfrak{A}$ is, except for sets of measure zero, equal to some E_γ; however we have no assurance that the sequence will be countable. As before we let $b_\gamma = \mu(E_\gamma)$ and $\bar{f}_\gamma$ be the value of f almost everywhere on E_γ. We need also a known lemma about the set in which a function in $L^p_\mu(Y)$ is not equal to 0.

LEMMA 8. *For every $q > 0$ and any $f \varepsilon L^q_\mu(Y)$, $E_0 = E_y[f(y) \neq 0] \varepsilon \mathfrak{B}$.*

PROOF. Since $f \varepsilon L^q$ means $|f|^q \varepsilon L^1$, we need only consider $L^1_\mu(Y)$ and may suppose that $f(y) > 0$ for all $y \varepsilon Y$. By definition of the integral (see Saks, p. 19 ff.) $\int_Y f d\mu = \sup_g \int_Y g d\mu$ where the sup is taken over those functions g which are linear combinations of characteristic functions of sets of $\mathfrak{X}$ and which at the same time satisfy the inequalities $0 \leqq g(y) \leqq f(y)$ for almost all $y \varepsilon Y$. But such a function can be different from zero only on a set of finite measure if it is to have a finite integral. There is a sequence $\{g_i\}$ of these functions such that $g_i \uparrow f$ almost everywhere in Y and $\int_Y f d\mu = \lim_i \int_Y g_i d\mu$; hence the set where $f > 0$ is, except for a set of measure zero, the sum of the sets on which the $g_i > 0$, and each of these sets is of finite measure.

By this lemma it is possible to put each function f of $L_\mu^p(Y)$ into at least one class C_E such that $f(y) = 0$ in $Y - E$ and $E \varepsilon \mathfrak{B}$; then C_E is equivalent to $L_\mu^p(E)$, and, if we set $U_E(f) = U(f)$ for $f \varepsilon C_E$, U_E is linear on $L_\mu^p(E)$ and therefore can be expressed as before where the E_i of Theorem 4 will be those E_γ which, except for sets of measure zero, lie in E. But $f = 0$ on $Y - E$; so we can write $U(f) = \sum_\gamma u_\gamma \bar{f}_\gamma$ if it is not identically zero, with the convention that the sum of any number of terms in which $\bar{f}_\alpha$ or u_α is zero shall be zero. Since this can be done for each $f \varepsilon L_\mu^p(Y)$, we get our final result, including the previous theorems as special cases.

THEOREM 5. *A functional U on $L_\mu^p(Y)$ is linear if and only if* (a) $\mathfrak{A}$ *is empty and U is identically zero or* (b) $\mathfrak{A}$ *is not empty and U can be expressed in the form $U(f) = \sum_\gamma u_\gamma \bar{f}_\gamma$ and $\|U\| = \sup_\gamma |u_\gamma| b_\gamma^{-1/p}$.*

If we let L^{p*} be the space of linear functionals on $L_\mu^p(Y) = L^p$, it is clear that with the given norm L^{p*} is a Banach space. To make the study of its structure simpler we introduce the class Γ of ordinal numbers as follows; If $\mathfrak{A}$ is empty, Γ is also empty; if $\mathfrak{A}$ is not empty, Γ is the class of ordinals used in ordering the sets E_γ of $\mathfrak{A}$ defined above. We use $\mathbf{x} = \{x_\gamma\}$ to stand for a real-valued function on Γ and define the special function $\mathbf{x}_\gamma$ to have the value 1 at γ and 0 elsewhere. Let $\mathit{M} = \mathit{M}(\Gamma)$ be the class of bounded functions $\mathbf{x}$ with $\|\mathbf{x}\|_M = \sup_{\gamma \varepsilon \Gamma} |x_\gamma|$; for any $q > 0$ let $\mathit{L}^q = \mathit{L}^q(\Gamma)$ be the class of all $\mathbf{x}$ such that $\|\mathbf{x}\|_q = (\sum_{\gamma \varepsilon \Gamma} |x_\gamma|^q)^{1/q} < \infty$.

It is clear that the space $\mathit{L}^q(\Gamma)$ here defined is also the space $L_{\mu_0}^q(\Gamma)$ where μ_0 is the trivial measure assigning the measure 1 to each set containing just one point γ and $\mathfrak{X}$ is the smallest additive family on the countable sets in Γ. From Theorem 5 we have this result:

THEOREM 6. *If Γ is not empty, a functional U on L^p is linear if and only if it can be expressed as $U(\mathbf{x}) = \sum_{\gamma \varepsilon \Gamma} u_\gamma x_\gamma$ where $\|U\| = \sup_{\gamma \varepsilon \Gamma} |u_\gamma|$.*

Considering again the space L^{p*}, it is clear that when $\mathfrak{A}$ is not empty the transformation which associates the element $U \varepsilon L^{p*}$ defined by $U(f) = \sum_{\gamma \varepsilon \Gamma} u_\gamma \bar{f}_\gamma$ with the element $u = \{u_\gamma b_\gamma^{-1/p}\} \varepsilon \mathit{M}$ is one-to-one, linear both ways and norm-preserving between all of L^{p*} and all of M; but this is precisely the definition of equivalence of two Banach spaces (Banach, p. 180); hence:

(6) L^{p*} and M are equivalent Banach spaces.

This holds even for the case in which $\mathfrak{A}$ is empty if we define $M(\Gamma)$ for Γ empty to be the Banach space with just one element, the zero element.

Slightly modifying our previous usage, we let $f_{(\gamma)}$ be $b_\gamma^{-1/p}$ on E_γ and be zero elsewhere. If we extend the definition of equivalence of Banach spaces in the obvious way to these L^p spaces we get this result:

THEOREM 7. *$L_\mu^p(\sum_{\gamma \varepsilon \Gamma} E_\gamma)$ is equivalent to $L^p(\Gamma)$ under the linear transformation for which $Tf = \mathbf{x}$, where $\mathbf{x} = \{\mathbf{x}_\gamma\} = \{\bar{f}_\gamma b_\gamma^{1/p}\}$; that is, which takes $f_{(\gamma)}$ to $\mathbf{x}_\gamma$.*

PROOF. By linearity if $f = \sum_\gamma x_\gamma f_{(\gamma)}$, $Tf = \mathbf{x}$ must be $\sum_\gamma x_\gamma \mathbf{x}_\gamma$. Now $\|f\|_p = (\sum |x_\gamma|^p)^{1/p}$ and $\|\mathbf{x}\|_p = (\sum |x_\gamma|^p)^{1/p}$; so the transformation is one-to-one on all L^p and L^p and $\|Tf\| = \|f\|$ for all $f \varepsilon L^p(\sum_\gamma E_\gamma)$; so the spaces are equivalent.

COROLLARY. *If Γ' is any set of ordinals and $\Gamma' \subset \Gamma$, then $L^p(\Gamma')$ is equivalent to the subspace $L_\mu^p(\sum_{\gamma \varepsilon \Gamma'} E)$ of $L_\mu^p(Y)$.*

Since, as is well known, $L^{1*} = M$, the following relation between L^p and L^1 is not entirely unexpected.

(7) If we define on the subspace $L_\mu^p(\sum_{\gamma \varepsilon \Gamma} E_\gamma)$ of $L_\mu^p(Y)$ the transformation T by associating to $f = \sum_{\gamma \varepsilon \Gamma} x_\gamma f_{(\gamma)}$ the function $Tf = \mathbf{x} = \sum_\gamma x_\gamma \mathbf{x}_\gamma$, then T is linear on $L_\mu^p(\sum_{\gamma \varepsilon \Gamma} E_\gamma)$ to L^1 and $\|T\| = 1$.

In general, if Γ is not a finite set, the set of all Tf with $f \varepsilon L_\mu^p(\sum_{\gamma \varepsilon T} E_\gamma)$ is not all of L^1.

We conclude with some elementary remarks about the cases when Γ is a finite set; then the spaces $L^q(\Gamma)$, $q > 0$, can be gotten by giving a new pseudo-norm to an n-dimensional euclidean space. When $q \geqq 1$ this pseudo-norm is actually a norm and the "spheres" $E_x[\|\mathbf{x}\| \leqq \epsilon]$ are convex point sets, and we have some of the well known Minkowski spaces. When $0 < q < 1$ these sets are not convex but each contains an ordinary euclidean sphere and is contained in another such sphere, so the topology in all these $L^q(\Gamma)$ is the same as that in, say, $L^2(\Gamma)$, the space with euclidean metric. Convergence in $M(\Gamma)$ is also the same as that in $L^2(\Gamma)$ when Γ is finite. However these spaces are not in general equivalent as Banach spaces; if $\Gamma = \{1\}$, all are equivalent Banach spaces; if $\Gamma = \{1, 2\}$, $L^1(\Gamma)$ and $M(\Gamma)$ are equivalent; none of the others are.

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