A NEW APPROACH TO THE CRITICAL VALUE THEORY

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The well known inequalities of M. Morse have as algebraical foundation the rank-equations

$$\text{(I)}\qquad rB_i(\Sigma - \Sigma_1) = rB_i(\Sigma) - rB_i(\Sigma_1) + rD_i(\Sigma_1, \Sigma) + rD_{i-1}(\Sigma_1, \Sigma), \qquad i = 0, 1, 2, \cdots .$$

These formulas hold for any topological group-system Σ, any subsystem Σ_1 of Σ, and the difference-system $\Sigma - \Sigma_1$. (W. Mayer, *Topologische Gruppensysteme*, Monatshefte für Mathematik und Physik, vol. 47 (1938); henceforth referred to as M, TG.) Here $B_i(\Sigma)$ denotes the i-dimensional Betti group of Σ, while $B_i(\Sigma_1)$ and $B_i(\Sigma - \Sigma_1)$ are these groups for Σ_1 and $\Sigma - \Sigma_1$ respectively. The symbol $r(\ )$, of course, stands for the rank of the group in the parentheses. By $D_i(\Sigma_1, \Sigma)$ we mean the subgroup of $B_i(\Sigma_1)$ containing all the classes of this group whose elements bound in Σ.

The formula (I) was first derived for the case of a complex in Lefschetz' *Topology*, 1930 (p. 150), and independently for the complex modulo 2 by J. Rybarz, Monatshefte für Mathematik und Physik (1931).

In the generality needed here the proof of (I) is given in M, TG (pp. 54–57), under the assumption, of course, that all the ranks appearing in (I) are finite, since otherwise the formula would be meaningless. But the proof there given shows also that

(a) $rB_i(\Sigma - \Sigma_1) = \infty$ implies that either $rB_i(\Sigma)$ or $rD_{i-1}(\Sigma_1, \Sigma)$, or both, are infinite;

(b) $rB_i(\Sigma - \Sigma_1)$ finite implies $rD_{i-1}(\Sigma_1, \Sigma)$ finite, and if in addition $rB_i(\Sigma_1)$ is finite then $rB_i(\Sigma)$ is finite too; and

(c) $rB_i(\Sigma - \Sigma_1) = 0$ implies $rD_{i-1}(\Sigma_1, \Sigma) = 0$ and if in addition $rB_i(\Sigma_1)$ is finite, then $rB_i(\Sigma_1) = rB_i(\Sigma) + rD_i(\Sigma_1, \Sigma)$.

As an immediate consequence of equations (I) we notice the inequality

$$\text{(I}'\text{)}\qquad rB_i(\Sigma) \leqq rB_i(\Sigma_1) + rB_i(\Sigma - \Sigma_1),$$

which is true whenever the terms on the right are finite (remark (b)) and trivial otherwise. The next step in attaining the Morse inequalities is the application of (I) to $m+2$ topological group-systems satisfying the inclusion relations

$$\text{(1)}\qquad \Sigma_m \supset \Sigma_{m-1} \supset \cdots \supset \Sigma_0 \supset \Sigma_{-1}$$

with Σ_{-1} empty (all $L_i(\Sigma_{-1})$ being zero-groups). Assuming that all ranks involved are finite, we get by summation on k in

$$rB_i(\Sigma_k - \Sigma_{k-1}) = rB_i(\Sigma_k) - rB_i(\Sigma_{k-1}) + rD_i(\Sigma_{k-1}, \Sigma_k) + rD_{i-1}(\Sigma_{k-1}, \Sigma_k) \tag{2}$$

the formula

$$\sum_{k=0}^{m} rB_i(\Sigma_k - \Sigma_{k-1}) = rB_i(\Sigma_m) + \sum_{k=1}^{m} rD_i(\Sigma_{k-1}, \Sigma_k) + rD_{i-1}(\Sigma_{k-1}, \Sigma_k), \tag{3}$$

since all the $rB_i(\Sigma_{-1})$, $rD_i(\Sigma_{-1}, \Sigma_0)$ are zero, Σ_{-1} being empty.

Let Σ_m be a neighborhood-space and let Σ_k, $k=m-1, \cdots, -1$, be subspaces of Σ_m satisfying the inclusion relation (1). With the introduction of singular simplices and chains modulo 2 each space Σ_k gives rise to a topological group system, which we also denote by Σ_k. (The i-dimensional complexes K^i are the finite chains of singular simplices.) The relations (1) are then inclusion relations for the so-constructed group-systems Σ_k, and for these systems relation (3) will hold provided the ranks involved are finite. The finiteness of all these ranks will follow from the finiteness of those of the left side of (3), that is, of the

$$rB_i(\Sigma_k - \Sigma_{k-1}).$$

If these are finite, we see from (2) that $rB_i(\Sigma_0)$ $(=rB_i(\Sigma_0 - \Sigma_1))$ is finite, and thus from (2) and remark (b) we conclude that $rB_i(\Sigma_1)$ is finite. So, step-by-step, using (2) and remark (b) we find that all the ranks $rB_i(\Sigma_k)$ are finite. Since $D_i(\Sigma_{k-1}, \Sigma_k) \subset B_i(\Sigma_{k-1})$, the ranks of the groups D_i are finite too. We have then the result:

The equations (3) *hold for the group-systems* Σ_k *if only the ranks appearing on the left sides of these equations are finite.*

It is an interesting fact, in view of its geometrical implications, that the finiteness of the ranks $rB_i(\Sigma_k - \Sigma_{k-1})$ has as a consequence the finiteness of the Betti numbers $rB_i(\Sigma_k)$. As an additional remark it may be noticed that equations (3) can be written in the form of Morse's inequalities (of the strong type) if (3) holds for all indices i. Denoting by M_i the left sides of (3):

$$M_i = \sum_{k=0}^{m} rB_i(\Sigma_k - \Sigma_{k-1}), \tag{4}$$

and successively eliminating the sums with terms rD_i, we get

$$\begin{aligned} &M_0 \geqq R_0, \\ &M_1 - M_0 \geqq R_1 - R_0, \\ &M_2 - M_1 + M_0 \geqq R_2 - R_1 + R_0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ &M_n - M_{n-1} + \cdots \pm M_0 = R_n - R_{n-1} + \cdots \pm R_0, \end{aligned} \tag{3'}$$

where n is the dimension of the topological system Σ_m and R_i stands for $rB_i(\Sigma_m)$, the ith Betti number of Σ_m. The inequalities (3′), however, are not fully equivalent to equations (3) since they result only if (3) holds for all the dimensions i from zero up to n, the dimension of Σ_m. In addition, in (3) the difference

$$(M_i - M_{i-1} + \cdots \pm M_0) - (R_i - R_{i-1} + \cdots \pm R_0)$$

is shown to be $\sum rD_i(\Sigma_{k-1}, \Sigma_k)$, which fact, of course, is of importance since it readily yields the conditions for the equality sign in (3′).

A subdivision of the space Σ_m by subspaces Σ_k of Σ_m can be obtained by the use of a real bounded function $\mathfrak{J}$ defined on Σ_m. Let σ_0 and σ_m be the greatest lower and least upper bound of $\mathfrak{J}$ respectively, so that

$$\sigma_m \geqq \mathfrak{J} \geqq \sigma_0; \tag{5}$$

then intermediate values σ_k, $k=1, \cdots, m-1$, and $\sigma_{-1}<\sigma_0$ may be introduced and subjected to the inequalities

$$\sigma_m > \sigma_{m-1} > \cdots > \sigma_1 > \sigma_0 > \sigma_{-1} \tag{6}$$

if $\mathfrak{J}$ is not constant on Σ_m. If $\mathfrak{J}$ is constant, (6) can be reduced to $\sigma_0 > \sigma_{-1}$.

We now define the subspace Σ_k as the set of all points P of Σ_m such that $\mathfrak{J}(P) \leqq \sigma_k$, that is, symbolically,

$$\Sigma_k \equiv \{\mathfrak{J} \leqq \sigma_k\}. \tag{7}$$

(We observe that in this notation the subspace Σ_m defined in (7) coincides with the space originally so denoted.)

Constructing the corresponding topological group-systems, denoted likewise by Σ_m (with Σ_{-1} consisting of zero groups only), we see that equations (3) hold if only the sum M_i of (4) is finite. This shows the full generality of these equations, in which only the concepts of neighborhood space and its partition by subspaces (not necessarily defined by a function $\mathfrak{J}$) are used. (Later on the validity of formula

(3) will be shown for a partition of the space by a countable set of subspaces.)

Let Σ_m be a compact Riemannian manifold and $\mathfrak{F}$ a function of class C^2 defined on Σ_m and having a finite number only of stationary points. Corresponding to these points there are a finite number of stationary values σ_k, $k=0, 1, \cdots, m$, with σ_0 and σ_m the absolute minimum and maximum respectively. Using these values in defining the Σ_k as previously described and denoting by the same letters Σ_k the corresponding group-systems, we again arrive at equations (3), given the finiteness of their left sides M_i.

Denote by $\Sigma_k^{\cdot}$ the space of all points P of Σ_m with the property $\mathfrak{F}(P) < \sigma_k$; then obviously $\Sigma_k^{\cdot} \subset \Sigma_k$.

By definition (Seifert-Threlfall, *Variationsrechnung im Grossen*, §4; we refer to this henceforth as S.T.) the type numbers $m_i(\sigma)$ for any value σ of $\mathfrak{F}$ are the ranks of the Betti groups of $\Sigma_{(\sigma)} - \Sigma_{(\sigma)}^{\cdot}$, where $\Sigma_{(\sigma)}$ and $\Sigma_{(\sigma)}^{\cdot}$ are the point sets $\{\mathfrak{F} \leqq \sigma\}$ and $\{\mathfrak{F} < \sigma\}$ respectively. The value σ is called critical if some of the $m_i(\sigma)$ are different from zero.

Only a stationary value (that is, a value belonging to a stationary point) can be critical, so that only for the stationary values σ_k, $k=0, 1, \cdots, m$, are the type numbers

$$m_i(\sigma_k) = rB_i(\Sigma_k - \Sigma_k^{\cdot}) \tag{8}$$

not all necessarily zero. In consequence of our assumption all the stationary points are *isolated* and thus their contribution to the corresponding $m_i(\sigma_k)$ will be finite (S.T., §10). Since only a finite number of stationary points belong to a stationary value σ_k, *$m_i(\sigma_k)$ will be finite* (S.T., p. 87).

If therefore we can prove that

$$rB_i(\Sigma_k - \Sigma_k^{\cdot}) = rB_i(\Sigma_k - \Sigma_{k-1}), \tag{9}$$

then not only is (3) verified, but its left side is shown to be the sum of all the type numbers of dimension i. Thus, in proving isomorphisms between the Betti groups of the systems

$$\Sigma_k - \Sigma_k^{\cdot} \tag{α_k}$$

and

$$\Sigma_k - \Sigma_{k-1} \tag{β_k}$$

for $k=0, 1, \cdots, m$, we likewise prove (9) and thereby (3).

The above isomorphisms are included in the following two statements:

(a) Each class of $B_i(\Sigma_k - \Sigma_{k-1})$ lies in a definite class of $B_i(\Sigma_k - \Sigma_k^{\cdot})$.

(b) Each class of $B_i(\Sigma_k - \Sigma_k^{\cdot})$ contains one and only one class of $B_i(\Sigma_k - \Sigma_{k-1})$.

Indeed, (a) and (b) establish a correspondence between the classes of the above groups which is obviously an isomorphism.

As to the proof of the statements, (a) is almost self-evident, since an i-cycle of $\Sigma_k - \Sigma_{k-1}$ is an i-cycle of $\Sigma_k - \Sigma_k^{\cdot}$, and an i-cycle of $\Sigma_k - \Sigma_{k-1}$ which is homologous to zero in $\Sigma_k - \Sigma_{k-1}$ is easily seen to be an i-cycle of $\Sigma_k - \Sigma_k^{\cdot}$ homologous to zero in $\Sigma_k - \Sigma_k^{\cdot}$.

We now prove the first part of (b), that is, each class of $B_i(\Sigma_k - \Sigma_k^{\cdot})$ contains a class of $B_i(\Sigma_k - \Sigma_{k-1})$. This is by no means evident, and to prove it we must make use of $\mathfrak{J}$-deformations, which in the case we are considering will exist. Suppose given $\{Z^i\} \subset B_i(\Sigma_k - \Sigma_k^{\cdot})$; then there exists a $K^{i-1} \subset \Sigma_k^{\cdot}$ such that

$$R(Z^i) = K^{i-1}. \tag{10}$$

But, on the compact point set K^{i-1}, the continuous function $\mathfrak{J}$ has somewhere a maximum which, of course, is smaller than σ_k, $(K^{i-1} \subset \Sigma_k^{\cdot})$. Thus K^{i-1} lies in some Σ^* defined by $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$, where $\epsilon > 0$ may be so chosen that $\sigma_k - \epsilon > \sigma_{k-1}$. Then (S.T., p. 87) an $\mathfrak{J}$-deformation exists which carries the point set $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$ into the point set $\{\mathfrak{J} \leqq \sigma_{k-1}\}$. Thus there exist a K^i on $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$ and a K_1^{i-1} on $\{\mathfrak{J} \leqq \sigma_{k-1}\}$ such that

$$R(K^i) = K^{i-1} + K_1^{i-1}. \tag{11}$$

Adding (10) to (11) we get

$$R(Z^i + K^i) = K_1^{i-1}, \tag{12}$$

thus showing that $Z^i + K^i$, a cycle of the class $\{Z^i\}$ of $B_i(\Sigma_k - \Sigma_k^{\cdot})$, is a cycle of $\Sigma_k - \Sigma_{k-1}$. Hence $\{Z^i\}$ contains this cycle of $\Sigma_k - \Sigma_{k-1}$, and, according to (a), the class of $B_i(\Sigma_k - \Sigma_{k-1})$ it represents.

We finally prove the second part of (b), namely: Each class of $B_i(\Sigma_k - \Sigma_k^{\cdot})$ contains only one class of $B_i(\Sigma_k - \Sigma_{k-1})$. This again follows from the statement: If a cycle Z^i of $\Sigma_k - \Sigma_{k-1}$ considered as a cycle of $\Sigma_k - \Sigma_k^{\cdot}$ is homologous to zero in $\Sigma_k - \Sigma_k^{\cdot}$, then this cycle will be homologous to zero in $\Sigma_k - \Sigma_{k-1}$. For this proof one must again make use of the $\mathfrak{J}$-deformation mentioned above. Let Z^i be a cycle of $\Sigma_k - \Sigma_{k-1}$ homologous to zero in $\Sigma_k - \Sigma_k^{\cdot}$; then there exist a K^{i+1} in Σ_k and a K^i in $\Sigma_k^{\cdot}$ such that

$$R(K^{i+1}) = Z^i + K^i, \tag{13}$$

which shows that $Z^i \sim 0$ in $\Sigma_k - \Sigma_k^{\cdot}$. Furthermore there will exist a K^{i-1} in Σ_{k-1} such that

$$(14) \qquad R(Z^i) = K^{i-1},$$

since Z^i is a cycle of $\Sigma_k - \Sigma_{k-1}$. From (13) and (14) follows

$$(15) \qquad R(K^i) = K^{i-1}.$$

The compact set K^i of Σ_k, $(\mathfrak{J} < \sigma_k)$, will lie in some $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$. Using the $\mathfrak{J}$-deformation carrying $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$ into $\{\mathfrak{J} \leqq \sigma_{k-1}\}$, we establish the existence of a chain K_1^{i+1} of $\{\mathfrak{J} \leqq \sigma_k - \epsilon\}$ and two chains K_1^i and L_1^i of $\{\mathfrak{J} \leqq \sigma_{k-1}\}$ satisfying the relation

$$(16) \qquad R(K_1^{i+1}) = K^i + K_1^i + L_1^i,$$

where L_1^i, as the deformation chain of $R(K^i) = K^{i-1}$ (which lies in Σ_{k-1}) will lie in Σ_{k-1}, by definition of an $\mathfrak{J}$-deformation. Adding the two relations (13) and (16), we get finally

$$(17) \qquad R(K^{i+1} + K_1^{i+1}) = Z^i + K_1^i + L_1^i,$$

where $K_1^i + L_1^i \subset \Sigma_{k-1}$ and $K^{i+1} + K_1^{i+1} \subset \Sigma_k$. This proves that $Z^i \sim 0$ in $\Sigma_k - \Sigma_{k-1}$.

We add that in proving (9) we made no use of the fact that either of the values σ_k or σ_{k-1} was stationary. We used only the existence of the above mentioned $\mathfrak{J}$-deformations, and these will always exist provided only that between σ_k and σ_{k-1} there are no stationary values. Given this condition, (9) always holds (of course, in the form $0=0$ when σ_k is not a stationary value).

The theory so far developed is unable to deal with the concept of the "variational calculus in the large," since in this case the function $\mathfrak{J}$ defined on Σ is unbounded above. We now adapt the theory to that case.

First we prove the following lemma:

LEMMA. *For three group-systems Σ_α, $\alpha = 1, 2, 3$, satisfying the inclusion relations*

$$(18) \qquad \Sigma_1 \subset \Sigma_2 \subset \Sigma_3,$$

the relation

$$(19) \qquad rD_i(\Sigma_1, \Sigma_3) \leqq rD_i(\Sigma_1, \Sigma_2) + rD_i(\Sigma_2, \Sigma_3)$$

holds.

PROOF. Each class of $D_i(\Sigma_1, \Sigma_3)$ is contained in a definite class of $D_i(\Sigma_2, \Sigma_3)$ since each cycle of Σ_1 bounding in Σ_3 is a cycle of Σ_2 bounding in Σ_3 and each cycle of Σ_1 bounding in Σ_1 is a cycle of Σ_2 bounding in Σ_2. By correlating with an element of $D_i(\Sigma_1, \Sigma_3)$ that element of $D_i(\Sigma_2, \Sigma_3)$ which contains it, we define a homomorphism,

$$D_i(\Sigma_1, \Sigma_3) \rightarrow D_i(\Sigma_2, \Sigma_3), \tag{20}$$

whose kernel group is the subgroup of $D_i(\Sigma_1, \Sigma_3)$ consisting of all classes bounding in Σ_2, hence $D_i(\Sigma_1, \Sigma_2)$. Denoting the map-group of the homomorphism by $D_i'(\Sigma_2, \Sigma_3)$ we have the isomorphism

$$D_i(\Sigma_1, \Sigma_3) - D_i(\Sigma_1, \Sigma_2) \approx D_i'(\Sigma_2, \Sigma_3) \tag{21}$$

and the rank-equation

$$rD_i(\Sigma_1, \Sigma_3) = rD_i(\Sigma_1, \Sigma_2) + rD_i'(\Sigma_2, \Sigma_3), \tag{22}$$

whence (19) follows (since $D_i'(\Sigma_2, \Sigma_3) \subset D_i(\Sigma_2, \Sigma_3)$).

Combining (I) and (19) we derive

$$rB_i(\Sigma_3 - \Sigma_1) \leqq rB_i(\Sigma_3 - \Sigma_2) + rB_i(\Sigma_2 - \Sigma_1), \tag{23}$$

thus showing that in subdividing a given partition we never decrease the sum M_i.

As for the validity of the proof given for (23), we have to establish the legality of the use of formula (I). This use will indeed be legitimate if the rank equations (I) hold for $B_i(\Sigma_3 - \Sigma_2)$ and $B_i(\Sigma_2 - \Sigma_1)$, since then all ranks in (I) for $\Sigma = \Sigma_3$ are finite, by remark (a) of page 838 and equation (19), and the employment of (I) is thus justified for all the three Betti groups appearing in (23). This condition will in fact be satisfied whenever (23) is used in the sequel.

Equation (23), however, is true without restrictions, as we show by the use of a formula of a more general type, namely, (4′) of page 46 in M, TG. By this formula, (18) implies the isomorphism $\Sigma_3 - \Sigma_2 \approx (\Sigma_3 - \Sigma_1) - (\Sigma_2 - \Sigma_1)$ which generalizes a well known group-isomorphism to group-systems. Using (I′) for the replacement of Σ and Σ_1 by $\Sigma_3 - \Sigma_1$ and $\Sigma_2 - \Sigma_1$ respectively we again arrive at formula (23), but now without any restriction.

After this remark we extend the proof of formula (3). Let the space Σ be subdivided by a countable set of subspaces Σ_k, $k = -1, 0, 1, \cdots$, satisfying the inclusion relation

$$\Sigma_{-1} \subset \Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_m \subset \cdots \subset \Sigma \tag{24}$$

with Σ_{-1} empty and $\sum_{m=0}^{\infty}(\Sigma_m) = \Sigma$ (we write $\Sigma = \Sigma_\infty$). Then formula (3) holds for an infinite but bounded sum

$$M_i = \sum_{k=1}^{\infty} rB_i(\Sigma_k - \Sigma_{k-1}) \tag{25}$$

if only the partition satisfies an additional assumption on its topological nature, namely:

(C) *Any compact point set of* Σ *lies in some* Σ_k $(k \neq \infty)$.

As a matter of fact, (C) is satisfied if the partition (24) is defined by the level surfaces of a continuous real function $\mathfrak{J}$ unbounded above, the subspace Σ_k being the set of all points P such that $\mathfrak{J}(P) \leqq \sigma_k$ and $\sigma_{-1} < \sigma_0 < \sigma_1 < \cdots$ being an unbounded sequence, with $\sigma_{-1} < \mathfrak{J}(P)$ for all points P of Σ.

Since by assumption M_i is finite, the left side of (3) will be finite, and thus (3) holds for any m. Thus each sum

$$\sum_{k=1}^{m} rD_i(\Sigma_{k-1}, \Sigma_k), \qquad \sum_{k=1}^{m} rD_{i-1}(\Sigma_{k-1}, \Sigma_k) \tag{26}$$

in (3), being not larger than M_i, converges as $m \to \infty$. As a further consequence, $rB_i(\Sigma_m)$ also converges as $m \to \infty$:

$$\lim_{m\to\infty} rB_i(\Sigma_m) = R_i'. \tag{27}$$

If only we can show that $R_i' = rB_i(\Sigma)$, we shall have proved formula (3) for $m = \infty$, that is,

$$M_i = R_i + \sum_{k=1}^{\infty} rD_i(\Sigma_{k-1}, \Sigma_k) + \sum_{k=1}^{\infty} rD_{i-1}(\Sigma_{k-1}, \Sigma_k), \tag{28}$$

writing $R_i = rB_i(\Sigma)$.

Due to the convergence of the sequences (26) and (27), all of whose terms—as ranks—are positive integers, or zeros, there exists an integer N such that for $k \geqq N$

$$\begin{aligned} rB_i(\Sigma_k - \Sigma_{k-1}) &= 0, \qquad & rD_i(\Sigma_{k-1}, \Sigma_k) &= 0, \\ rD_{i-1}(\Sigma_{k-1}, \Sigma_k) &= 0, \qquad & rB_i(\Sigma_k) - R_i' &= 0. \end{aligned} \tag{29}$$

Consequently, for $h > k > N$ the right-hand terms of the inequality

$$rB_i(\Sigma_h - \Sigma_k) \leqq \sum_{j=k+1}^{h} rB_i(\Sigma_j - \Sigma_{j-1}) \tag{30}$$

are zero, so that

$$rB_i(\Sigma_h - \Sigma_k) = 0. \tag{31}$$

From (31) and the assumption (C) we easily conclude that for $k > N$

$$rB_i(\Sigma - \Sigma_k) = 0. \tag{32}$$

In fact, (32) is proved if, for any C^i representing an element of $B_i(\Sigma-\Sigma_k)$, we can show that $C^i\sim 0$ in $\Sigma-\Sigma_k$. As a compact set on Σ, ($C^i\subset\Sigma$ with $R(C^i)\subset\Sigma_k$), C^i lies in some $\Sigma_h\supset\Sigma_k$, and thus represents an element of $B_i(\Sigma_h-\Sigma_k)$. But since this group has the rank zero, by (31), $C^i\sim 0$ in $\Sigma_h-\Sigma_k$. Therefore there exists a K^{i+1} and a K^i satisfying

$$K^{i+1}\subset\Sigma_h, \qquad K^i\subset\Sigma_k, \tag{33}$$

such that

$$R(K^{i+1}) = C^i + K^i; \tag{34}$$

this shows that $C^i\sim 0$ in $\Sigma-\Sigma_k$ and proves (32).

Using (32) in connection with the main formula (I) we have for $k>N$

$$0 = rB_i(\Sigma) - rB_i(\Sigma_k) + rD_i(\Sigma_k, \Sigma) + rD_{i-1}(\Sigma_k, \Sigma). \tag{35}$$

Remark. The use of (I) is again justified, since all ranks involved are finite. (For $rB_i(\Sigma)$ and $rD_{i-1}(\Sigma_k, \Sigma)$ see remark (b) on page 838; remember also that $rB_i(\Sigma_k)=R_i^{\prime}$ and $rD_i(\Sigma_k, \Sigma)\leqq rB_i(\Sigma_k)$.) According to our remark (c) on page 838 we have $rD_{i-1}(\Sigma_k, \Sigma)=0$ in formula (35). *We shall prove that in addition* $rD_i(\Sigma_k, \Sigma)=0$.

This again is shown if for any C^i representing an element of $D_i(\Sigma_k, \Sigma)$ we can prove that $C^i\sim 0$ in Σ_k. Let C^i be such an element; then there exists a K^{i+1} such that

$$C^i=R(K^{i+1}), \qquad C^i\subset\Sigma_k. \tag{36}$$

But, again, the compact set K^{i+1} lies in some $\Sigma_h\supset\Sigma_k$, and thus C^i represents an element of $D_i(\Sigma_k, \Sigma_h)$, whose rank is not larger than

$$\sum_{j=k}^{h-1} rD_i(\Sigma_j, \Sigma_{j+1}) = 0.$$

Therefore $rD_i(\Sigma_k, \Sigma_h)=0$, so that C^i is homologous to zero in Σ_k, which was to be proved.

This shown, (35) reduces to

$$rB_i(\Sigma_k) = rB_i(\Sigma), \qquad k > N, \tag{37}$$

thus proving (28).

If for the concept of variational calculus in the large ($f(x, x')$ positive definite and of class C^2) we restrict ourselves to the case where only a finite number of stationary points (extremals) are below any $\mathfrak{J}$-level ($\mathfrak{J}=\int_P^Q f(x, x')dt>0$) and if the sequence σ_k, $k=0, 1, \cdots$, unbounded above, includes all the stationary values, then $\mathfrak{Z}$-deformations exist (S.T., p. 66), and can be used to prove homomorphisms between the systems (α_k) and (β_k) on page 841 in exactly the same way as before.

Hence (9) will hold, and again the type numbers

$$m_i(\sigma_k) = rB_i(\Sigma_k - \dot{\Sigma_k}) \tag{38}$$

are finite (S.T., §14, Theorem II and §17, Theorem II), so that (3) is true for any m.

In taking the sum $M_i = \Sigma m_i(\sigma_k)$, only critical values σ_k (by their definition) add nonzero terms, and if noncritical values σ were used in the construction, they can be omitted in writing M_i. Hence for this case too the Morse equations hold for a finite M_i.

Remark. If, in the case just considered, for any two subspaces Σ', Σ'' of Σ the ranks of $D_i(\Sigma', \Sigma'')$ and $D_{i-1}(\Sigma', \Sigma'')$ are zero $(\Sigma' \subset \Sigma'')$, then $M_i = R_i$. For finite M_i, of course, this follows from (28), but for infinite M_i we have to prove the statement. First, from (I)—or more exactly from the proof leading to (I)—we have, for any[1] k

$$rB_i(\Sigma) \geqq rB_i(\Sigma_k); \tag{39}$$

then for any m we have

$$\sum_{k=0}^{m} rB_i(\Sigma_k - \Sigma_{k-1}) = rB_i(\Sigma_m). \tag{40}$$

Since the left side of (40) diverges for $m \to \infty$, so does $rB_i(\Sigma_m)$ diverge, and hence, by (39), $rB_i(\Sigma)$ cannot be finite.

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[1] In fact, we have $B_i(\Sigma - \Sigma_k) \approx B_i(\Sigma) - F_i(\Sigma, \Sigma_k)$, $F_i(\Sigma, \Sigma_k) \approx B_i(\Sigma_1)$ (M, TG, pp. 55–57).