REMARKS ON A NOTE OF MR. R. WILSON AND ON RELATED SUBJECTS[1]

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Introduction. *Let $w(x)$ be a nonnegative weight function on the interval $-1 \leqq x \leqq +1$, and let the integral*

$$(1) \qquad \int_{-1}^{+1} \log w(x) \cdot (1 - x^2)^{-1/2} dx = \int_0^{\pi} \log w(\cos \theta) d\theta$$

exist in the sense of Lebesgue.

If $\{p_n(x) = k_n x^n + \cdots\}$ denotes the orthonormal set of polynomials associated with $w(x)$, we have

$$(2) \qquad \lim_{n \to \infty} \max_{-1 \leqq x \leqq +1} | p_n(x) |^{1/n} = 1,$$

and[2]

$$(3) \qquad \lim_{n \to \infty} k_n^{1/n} = 2.$$

In 1921 I found[3] the following asymptotic formula for the orthogonal polynomials $p_n(x)$ for $n \to \infty$, holding for x not on the segment $[-1, +1]$:

$$(4) \qquad \lim_{n \to \infty} z^n p_n(x) = \Delta(z)$$

where $2x = z + z^{-1}$, $|z| < 1$, and $\Delta(z)$ is a certain analytic function regular and nonzero for $|z| < 1$. Of course, $\Delta(z)$ depends on the weight function $w(x)$. The formula (4) holds uniformly for

$$|z| \leqq r, \; r < 1.$$

From this result the formulas (3) and, by an additional elementary remark (cf. below (9)), (2) follow immediately. Also it furnishes (cf. OP, p. 302, Theorem 12.7.1):

$$(5) \qquad \lim_{n \to \infty} 2^{-n} k_n = \pi^{-1/2} \exp \left\{ -\frac{1}{2\pi} \int_0^{\pi} \log w(\cos \theta) d\theta \right\}.$$

[1] Presented to the Society, February 24, 1940.

[2] Concerning the notation see my book *Orthogonal Polynomials* (American Mathematical Society Colloquium Publications, vol. 23, 1939). Hereafter this book will be referred to as OP.

[3] G. Szegö, *Über die Entwickelung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems*, Mathematische Annalen, vol. 82 (1921), pp. 188–212; p. 191. Cf. also OP, p. 290, Theorem 12.1.2.

Mr. R. Wilson devoted a recent note[4] to the proof of the relation (2), or rather of the following relation:

$$\lim_{n\to\infty} \max_{-1\leqq x\leqq 1} \left| k_n^{-1} p_n(x) \right|^{1/n} = \tfrac{1}{2}, \tag{6}$$

which is, on account of (3), equivalent to (2). His argument is based on certain results of Mr. Shohat,[5] which are incidentally consequences of the asymptotic formula (4). The conditions used by Shohat are more restrictive than the existence of (1).

By applying a classical theorem of Poincaré on recurrence formulas, Shohat proves that

$$\lim_{n\to\infty} \left| p_n(x) \right|^{1/n} = \left| z \right|^{-1}, \tag{7}$$

where x and z have the same meaning as before and $|z| < 1$. (From this, (3) follows for $x = \infty$ or $z = 0$.) Based on (7), a proof of (2) or (6) can easily be arranged. Wilson (loc. cit., p. 191) prefers, however, to use another theorem on recurrence formulas due to Perron.[6]

In the present note, I give first a very simple direct approach to (2), (3), and (6), assuming the existence of (1). Naturally the deeper result (5) requires more refined methods.

Further, we deal with the following related result of Shohat (loc. cit., pp. 34–36): *Let $w(x) \geqq 0$ be an arbitrary weight function on the interval $-1 \leqq x \leqq +1$. Employing the former notation, the relations*

$$2^{-n} k_n = O(1), \qquad \lim_{n\to\infty} 2^{-n} k_n \text{ exists} \tag{8}$$

are equivalent. Of course, this means that the second relation follows from the first one. The proof, given below, is essentially Shohat's argument; we found it, however, convenient and possible to eliminate every reference to the theory of continued fractions used by Shohat.

Finally, by means of the deeper result (5) we show that the existence of the integrals (1) is not only *sufficient* but also *necessary* for the relations (8). More precisely: *Let $w(x)$ be a nonnegative weight*

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[4] R. Wilson, *A note on the asymptotic properties of orthogonal polynomials*, this Bulletin, vol. 45 (1939), pp. 190–192.

[5] See J. Chokhatte (Shohat), *Sur le développement de l'intégrale $\int_a^b [p(y)/(x-y)]dy$ en fraction continue et sur les polynômes de Tchebycheff*, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 25–46; cf. in particular pp. 43–44.

[6] The following objection can be made to his argument. In the present case the coefficients of the recurrence formula contain x as a parameter. However, max $|p_n(x)|$, $-1 \leqq x \leqq +1$, will be in general attained for an $x = x(n)$ which varies with n. Therefore, a generalization of Perron's theorem is needed here stating the *uniform* existence of the limit involved.

function on the interval $-1 \leqq x \leqq +1$, *and let* $p_n(x) = k_n x^n + \cdots$ *have the same meaning as before. If* $2^{-n}k_n = O(1)$, *the integrals* (1) *exist.*

Proof of (2), (3), (6). It is well known that

$$\max_{-1 \leqq x \leqq +1} \left| k_n^{-1} p_n(x) \right| \geqq 2^{1-n}, \tag{9}$$

so that for (6) it suffices to show that in $-1 \leqq x \leqq +1$

$$k_n^{-1} \left| p_n(x) \right| < A \cdot 2^{-n}(1 + \epsilon)^n; \tag{10}$$

here $\epsilon > 0$ is arbitrary and A depends only on $w(x)$ and ϵ.

Let $z = re^{i\phi}$, $0 \leqq r < 1$. By use of the inequality between the arithmetic and geometric mean of a function,[7] we find that

$$\begin{aligned}
\pi^{-1} &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} w(\cos\theta) \left|\sin\theta\right| \{p_n(\cos\theta)\}^2 d\theta \\
&\geqq \frac{1-r}{1+r} \frac{1}{2\pi} \int_{-\pi}^{+\pi} w(\cos\theta) \left|\sin\theta\right| \{p_n(\cos\theta)\}^2 \\
&\qquad\qquad \cdot \frac{1-r^2}{1 - 2r\cos(\theta - \phi) + r^2} d\theta \\
&\geqq \frac{1-r}{1+r} \cdot \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{+\pi} \log\left[w(\cos\theta)\left|\sin\theta\right|\right] \right. \\
&\qquad\qquad \left. \cdot \frac{1-r^2}{1 - 2r\cos(\theta - \phi) + r^2} d\theta\right\} \\
&\qquad \cdot \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{+\pi} \log\{p_n(\cos\theta)\}^2 \right. \\
&\qquad\qquad \left. \cdot \frac{1-r^2}{1 - 2r\cos(\theta - \phi) + r^2} d\theta\right\}.
\end{aligned} \tag{11}$$

The harmonic function

$$\log \left| z^n p_n\left(\tfrac{1}{2}(z + z^{-1})\right) \right|^2 = 2\Re \log \left\{ z^n p_n\left(\tfrac{1}{2}(z + z^{-1})\right) \right\} \tag{12}$$

is regular for $|z| < 1$, and it has logarithmic singularities at $z = e^{\pm i\alpha_\nu}$, if $\cos\alpha_\nu$, $\nu = 1, 2, \cdots, n$, denote the roots of $p_n(x)$. Therefore, the second exponential expression in (11) becomes

$$\exp\left\{\log\left| z^n p_n\left(\tfrac{1}{2}(z+z^{-1})\right)\right|^2\right\} = |z|^{2n} \left| p_n\left(\tfrac{1}{2}(z + z^{-1})\right)\right|^2.$$

Consequently,

[7] See, for instance, G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Berlin, 1925, pp. 54, 210, Problem 79.

$$
\begin{aligned}
(13)\qquad & |z|^n |p_n(\tfrac{1}{2}(z+z^{-1}))| < \pi^{-1/2}\left(\frac{1+r}{1-r}\right)^{1/2} \\
& \qquad \cdot \exp\left\{\frac{1}{2\pi}\int_0^\pi |\log [w(\cos\theta)\sin\theta]|\, d\theta\cdot\frac{1+r}{1-r}\right\}.
\end{aligned}
$$

For $z=0$ this results in the inequality

$$
2^{-n}k_n < \pi^{-1/2}\exp\left\{\frac{1}{2\pi}\int_0^\pi |\log [w(\cos\theta)\sin\theta]|\, d\theta\right\}.
$$

On the other hand, for $x=\cos\theta$,

$$
\begin{aligned}
k_n^{-2} &= \min \int_{-1}^{+1} w(x)(x^n+\cdots)^2 dx \\
&\leqq \int_{-1}^{+1} w(x)(2^{1-n}\cos n\theta)^2 dx < 2^{2-2n}\int_{-1}^{+1} w(x)dx,
\end{aligned}
$$

so that $2^{-n}k_n$ remains between two positive bounds. From here (3) follows. Choosing $|z|=1-\delta$ in (13) where $\delta>0$ is sufficiently small, and applying the maximum-principle in the corresponding ellipse of the x-plane, we obtain (10) which implies (6) and also (2).

Proof of the equivalence of the relations (8) **(Shohat's theorem).** Let (see OP, p. 41, Theorem 3.2.1)

$$
(14)\quad p_n(x) = k_n\bar{p}_n(x) = k_n(x^n - S_n x^{n-1}+\cdots),
$$

$$
(15)\quad \bar{p}_n(x) = (x-c_n)\bar{p}_{n-1}(x) - \left(\frac{k_{n-2}}{k_{n-1}}\right)^2 \bar{p}_{n-2}(x),
$$

$$
(16)\qquad b_{2n} = -\frac{k_{n-1}}{k_n}\frac{p_n(-1)}{p_{n-1}(-1)} > 0,\quad b_{2n+1} = -\frac{k_{n-1}}{k_n}\frac{p_{n-1}(-1)}{p_n(-1)} > 0,
$$

$$
n = 1, 2, 3, \cdots;\ \bar{p}_{-1}(x) = 0.
$$

Obviously,

$$
(17)\qquad k_0^{-2}k_n^2 = (b_2b_3\cdots b_{2n}b_{2n+1})^{-1},
$$

$$
(18)\quad c_n+1 = b_{2n}+b_{2n-1},\ S_n = \sum_{\nu=1}^{2n} b_\nu - n,\ \ n=1, 2, 3, \cdots;\ b_1 = 0.
$$

Since

$$
(19)\qquad c_n\int_{-1}^{+1}\{p_{n-1}(x)\}^2 w(x)dx = \int_{-1}^{+1} x\{p_{n-1}(x)\}^2 w(x)dx,
$$

we have $|c_n|<1$, so that from (18) $0<b_n<2$ follows (except $b_1=0$).

If $T_n(x)$ denotes the polynomial of Tchebichef, we write

$$T_n(x) = h_0p_0(x) + h_1p_1(x) + \cdots + h_np_n(x); \tag{20}$$

then

$$\begin{aligned} h_\nu &= \int_{-1}^{+1} T_n(x)p_\nu(x)w(x)dx, \\ |h_\nu| &\leqq \int_{-1}^{+1} |p_\nu(x)|\, w(x)dx \leqq \left\{\int_{-1}^{+1} w(x)dx\right\}^{1/2}, \end{aligned} \tag{21}$$

$$\nu = 0, 1, 2, \cdots, n.$$

Comparing the coefficients of x^n and x^{n-1} in (20), we find $2^{n-1} = h_nk_n$, $0 = -h_nk_nS_n + h_{n-1}k_{n-1}$, so that

$$S_n = 2^{1-n}h_{n-1}k_{n-1}, \qquad |S_n| \leqq 2^{1-n}k_{n-1}\left\{\int_{-1}^{+1} w(x)dx\right\}^{1/2}.$$

Now, we define $U_n = 2b_n - 1$, $-1 < U_n < 3$ (except $U_1 = -1$). Then, as $n \to \infty$,

$$\begin{aligned} \sum_{\nu=1}^{2n} U_\nu &= 2S_n = 2^{1-n}k_{n-1}\cdot O(1), \\ \prod_{\nu=2}^{2n+1}(1 + U_\nu)^{-1} &= \prod_{\nu=2}^{2n+1}(2b_\nu)^{-1} = 2^{-2n}k_0^{-2}k_n^2 = (2^{-n}k_n)^2\cdot O(1). \end{aligned} \tag{22}$$

Assuming $2^{-n}k_n = O(1)$, both expressions (22) remain bounded.

Let c be a positive constant such that

$$u^{-2}\{u - \log(1 + u)\} > c, \qquad -1 < u < 3. \tag{23}$$

Thus $cU_n^2 < U_n - \log(1 + U_n)$, so that $\sum_{\nu=1}^{2n} U_\nu^2 = O(1)$, that is, $\sum U_n^2$ is convergent. The same holds for

$$\begin{aligned} \sum_{n=1}^{\infty} U_{2n}U_{2n+1} &= \sum_{n=1}^{\infty}\{4b_{2n}b_{2n+1} - 2(b_{2n} + b_{2n+1}) + 1\} \\ &= \sum_{n=1}^{\infty}\{4b_{2n}b_{2n+1} - 2(b_{2n} + b_{2n-1}) + 1\} \\ &\quad + \sum_{n=1}^{\infty} 2(b_{2n-1} - b_{2n+1}) \\ &= \sum_{n=1}^{\infty}\left\{4\left(\frac{k_{n-1}}{k_n}\right)^2 - 2(S_n - S_{n-1}) - 1\right\} \\ &\quad + \sum_{n=1}^{\infty} 2(b_{2n-1} - b_{2n+1}). \end{aligned} \tag{24}$$

The last series is convergent since $U_n \to 0$ or $b_n \to \frac{1}{2}$. Therefore,

$$(25) \qquad \sum_{n=1}^{\infty} \left\{ \left(\frac{k_{n-1}}{k_n}\right)^2 - 2(S_n - S_{n-1}) - 1 \right\}$$

is convergent. Applying this result to the polynomials $(-1)^n p_n(-x) = k_n(x^n + S_n x^{n-1} + \cdots)$ associated with the weight function $w(-x)$ on $[-1, +1]$, we obtain the convergence of

$$(26) \qquad \sum_{n=1}^{\infty} \left\{ \left(\frac{k_{n-1}}{k_n}\right)^2 - 2(-S_n + S_{n-1}) - 1 \right\},$$

or that of $\sum_{n=1}^{\infty}(S_n - S_{n-1})$. This is equivalent to the existence of $\lim_{n\to\infty} S_n$, or to the convergence of $\sum U_n$. This, together with the convergence of $\sum U_n^2$, implies the convergence of the product $\prod(1+U_n)^{-1}$ or the existence of $\lim_{n\to\infty} 2^{-n}k_n$.

Proof of the equivalence of the conditions (8) **to the existence of the integrals** (1). The relations (8) are equivalent to the fact that for every polynomial $q(x) = x^n + \cdots$ of the nth degree with the highest term x^n

$$(27) \qquad \int_{-1}^{+1} \{q(x)\}^2 w(x)dx > c \cdot 2^{-2n}$$

holds, where $c > 0$ is independent of n. Let $\epsilon > 0$; we have

$$(28) \qquad \int_{-1}^{+1} \{q(x)\}^2 (w(x) + \epsilon)dx > c \cdot 2^{-2n}.$$

The minimum of the left-hand side is $\{k_n(\epsilon)\}^{-2}$, where $k_n(\epsilon)$ denotes the highest coefficient of the orthonormal polynomial of the nth degree associated with $w(x) + \epsilon$. For this weight function, the integral condition (1) is satisfied so that according to (5)

$$(29) \qquad \lim_{n\to\infty} 2^{-n}k_n(\epsilon) = \pi^{-1/2} \exp\left\{ -\frac{1}{2\pi}\int_0^{\pi} \log\,[w(\cos\theta) + \epsilon]d\theta \right\}.$$

Therefore,

$$\pi \exp\left\{ \pi^{-1} \int_0^{\pi} \log\,[w(\cos\theta) + \epsilon]d\theta \right\} \geqq c,$$

$$\int_0^{\pi} \log\,[w(\cos\theta) + \epsilon]d\theta \geqq c'$$

where c' is independent of ϵ. Now, $\log(\alpha+\beta)<\log\beta+1$ for $0<\alpha<\frac{1}{2}<\beta$; so if $0<\epsilon<\frac{1}{2}$,

$$\int_{w(x)>1/2}\log\left[w(\cos\theta)+\epsilon\right]d\theta<\int_{w(x)>1/2}\left\{\log w(\cos\theta)+1\right\}d\theta,$$

hence

$$\int_{w(x)\leqq 1/2}\log\left[w(\cos\theta)+\epsilon\right]d\theta>c''$$

where c'' is independent of ϵ. The same inequality holds if the integration is extended over the set $0<\eta\leqq w(x)\leqq\frac{1}{2}$. But for a decreasing sequence of bounded (negative) functions the operations of integration and passing to the limit as $\epsilon\rightarrow+0$ are interchangeable; consequently,

$$\int_{\eta\leqq w(x)\leqq 1/2}\log w(\cos\theta)d\theta\geqq c'',$$

and since this is true for all $\eta>0$, the integral (1) exists.

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