A NOTE ON MAXIMUM MODULUS AND THE ZEROS
OF AN INTEGRAL FUNCTION

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Let \( f(z) \) be an integral function of finite order \( \rho \geq 0 \), let \( M(r, f) = \max_{|z|=r} |f(z)| \), and let \( n(r, f) = n(r) \) be the number of zeros of \( f(z) \) in \( |z| \leq r \) and on its circumference. I have discussed elsewhere\(^1\) the behaviour of \( g(r) = \log M(r)/n(r) \), as \( r \to \infty \), and have proved that for every canonical product function \( f(z) \)

\[
\liminf_{r \to \infty} \frac{\log M(r)}{n(r)\phi(r)} = 0,
\]

where \( \phi(r) \) is any positive \( L \) function\(^2\) such that

\[
\int \frac{dx}{x\phi(x)} < A = \text{const.}
\]

The question arises how large and how small \( g(r) \) and \( G(r) = T(r, f)/n(r) \) can be, where \( T(r, f) = T(r) \) is the Nevanlinna characteristic function for \( f(z) \). I prove in this note the following result.

**Theorem.** Given \( \rho \geq 0 \) and \( \psi(x) \) any positive function such that

\[
\limsup_{x \to \infty} \frac{\log \psi(x)}{\log x} \leq \rho.
\]

There exists an integral function \( F(z) \) of order \( \rho \) for which

\[
\limsup_{r \to \infty} \frac{T(r, F)}{\psi(r)n(r, F)} = \infty
\]

and an integral function \( f(z) \) of order \( \rho \) for which

\[
\liminf_{r \to \infty} \frac{\psi(r)T(r, f)}{n(r, f)} = 0.
\]

**Proof.** We shall first construct an integral function \( f(z) \) of order \( \rho \) for which


\(^2\) For definition see G. H. Hardy, Orders of Infinity, 1924, p. 17.
\[(4.1) \quad \liminf_{r \to \infty} \frac{\log M(r, f) \psi(r)}{n(r, f)} = 0 \]

from which (4) will follow. Let

\[ \lambda_n = n^{\eta_n}, \quad \psi(x) = e^{(\rho + \eta_n)\log x} \]

where we may suppose without loss of generality that \( \eta_n > 0 \). We know that \( \eta_n \to 0 \) as \( x \to \infty \). Let \( \eta_r \) be the upper bound of \( \eta_i \) for \( i \geq r \). Then \( \eta_r \to 0 \) monotonically as \( r \to \infty \), and \( \psi(r) \leq e^{(\rho + \eta_r)\log r} \). Let \( \epsilon_1 = \epsilon_2 = 1 \), and

\[
\epsilon_n = 2\eta_n + \frac{(4 + \rho) \log \lambda_{n-1}}{\log \lambda_n}, \quad n = 3, 4, 5, \ldots ;
\]

\[ k = [\rho] + 1; \]

\[ \mu_n = \left[ \frac{\rho + \epsilon_n}{\lambda_n} \right], \quad \delta_n = \left[ \frac{\rho}{\lambda_n} + \epsilon_n \right], \quad n = 1, 2, 3, \ldots ; \]

\[ f(z) = \prod_{n=1}^{\infty} \left( 1 + \left( \frac{z}{\lambda_n} \right)^{k\mu_n} \right), \quad F(z) = \prod_{n=1}^{\infty} \left( 1 + \frac{z^k}{\lambda_n} \right)^{f_n} . \]

It is easily seen that \( f(z) \) and \( F(z) \) are canonical products. Further, \( n(\lambda_n, f) = k(\mu_1 + \cdots + \mu_n) \sim k\mu_n \) and hence

\[
\limsup_{r \to \infty} \frac{\log n(r, f)}{\log r} = \limsup_{n \to \infty} \frac{\log k + \log \mu_n}{\log \lambda_n} = \rho .
\]

Hence \( f(z) \) is an integral function of order \( \rho \). Let now \( x = \lambda_m \). Then

\[
\log f(x) = \sum_{n=1}^{m-1} \log \left\{ 1 + \left( \frac{\lambda_m}{\lambda_n} \right)^{k\mu_n} \right\} + \log 2 + \sum_{n=m+1}^{\infty} \log \left\{ 1 + \left( \frac{\lambda_m}{\lambda_n} \right)^{k\mu_n} \right\}
\]

\[
= \Sigma_1 + \log 2 + \Sigma_3 .
\]

We have

\[ \Sigma_3 \leq \sum_{n=m+1}^{\infty} \left( \frac{\lambda_m}{\lambda_n} \right)^{k\mu_n} \leq \sum_{n=m+1}^{\infty} \left( \frac{1}{2} \right)^{k\mu_n} = O(1) , \]

\[ \Sigma_1 < m \log \{ 2(\lambda_m)^{k\mu_{m-1}} \} . \]

Hence for \( m \geq m_0 \),

\[
\log f(x) < 2m \left\{ \log 2 + k\mu_{m-1} \log \lambda_m \right\} \leq \exp \left\{ \log \mu_{m-1} + 2 \log \log \lambda_m \right\} .
\]

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\[ \frac{\psi(x) \log f(x)}{n(x, f)} \leq \exp \left\{ (\rho + \eta_m) \log \lambda_m + \log \mu_{m-1} + 2 \log \log \lambda_m - \log \mu_m \right\} \]
\[ \leq \exp \left\{ (\rho + \eta_m) \log \lambda_m + \log \mu_{m-1} + 2 \log \log \lambda_m 
- (\rho + 2\eta_m + (4 + \rho) \log \lambda_{m-1}/\log \lambda_m) \log \lambda_m + O(1) \right\}, \]
and the last expression tends to zero with \( 1/m \). Hence
\[ \liminf_{r \to \infty} \frac{\psi(r) \log M(r, f)}{n(r, f)} = 0 \]
which proves (4.1). Further
\[ n(\lambda_n^{1/k}, F) = k(\zeta_1 + \cdots + \zeta_n) \sim k\zeta_n. \]
Hence \( F(z) \) is an integral function of order \( \rho \). Let now \( r = \frac{1}{2}\lambda_n^{1/k} \). Then
\[ n(r, F) = k(\zeta_1 + \cdots + \zeta_{n-1}) \leq k\zeta_{n-1} \left( 1 + \frac{2\zeta_{n-2}}{\zeta_{n-1}} \right) \]
\[ < \exp \left\{ \log k + \log \zeta_{n-1} + \frac{2\zeta_{n-2}}{\zeta_{n-1}} \right\}, \]
\[ \log M(\frac{3}{2}r, F) > \zeta_n \log \left( 1 + \frac{1}{3^k} \right) \]
\[ = A \exp \left\{ \left( \frac{\rho}{k} + \epsilon_n \right) \log \lambda_n + O(\lambda_n^{-\rho/k}) \right\}, \]
for \( n \geq n_0 \). Hencea
\[ \frac{\log M(\frac{3}{2}r, F)}{n(r, F)\psi(r)} > A \exp \left\{ \left( \frac{\rho}{k} + \epsilon_n \right) \log \lambda_n - \log \zeta_{n-1} \right. \]
\[ - (\rho + \eta_n) \left( \frac{1}{k} \log \lambda_n - \log 2 \right) + O(1) \right\} \]
\[ = A \exp \left\{ \epsilon_n \log \lambda_n - \log \zeta_{n-1} - \frac{\eta_n}{k} \log \lambda_n + O(1) \right\} \]
\[ = A \exp \left\{ 2\eta_n \log \lambda_n + (4 + \rho) \log \lambda_{n-1} \right. \]
\[ - \left( \frac{\rho}{k} + \epsilon_n \right) \log \lambda_{n-1} - \frac{\eta_n}{k} \log \lambda_n + O(1) \right\}, \]

a A denotes a positive constant.
and the last expression tends to infinity with $n$. Hence
\[\limsup_{r \to \infty} \frac{\log M(\varepsilon r, F)}{n(r, F)\psi(r)} = \infty,\]
and so
\[\limsup_{r \to \infty} \frac{T(r, F)}{\psi(r)n(r, F)} = \infty.\]

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