METRIC SEPARABILITY AND OUTER INTEGRALS[1]

JOHN F. RANDOLPH

R. L. Jeffery [1] investigated an upper integral for functions (from the line to real numbers) not necessarily measurable. Let f be bounded, $\alpha < f(x) < \beta$ for $x \varepsilon A$, let $e_i = E_x[a_i \leqq f(x) < a_{i+1}]$ where $\alpha = a_0 < a_1 < \cdots < a_n = \beta$ and consider $\sum a_i m^* e_i$. If as n increases and max $(a_{i+1} - a_i)$ approaches zero, the limit of this sum exists and is independent of the subdivisions, this limit is the upper integral of f over A, $\int_A^* f(x)dx$.

Two point sets with finite outer measure are metrically separable if the outer measure of their sum is the sum of their outer measures. A function is metrically separable on a set A if for each constant λ the part of A on which the function is greater than λ is metrically separable from the rest of A. Jeffery proved that metric separability may be made the basis of a comprehensive theory of integration which includes Young, Pierpont, and Lebesgue integration.

All measurable functions are metrically separable, but a function defined over a non-measurable set (and so necessarily non-measurable) may still be metrically separable. However, if f is metrically separable and possesses an outer integral on a set A, there exists a measurable set $B \supset A$ and a function ϕ, measurable on B and equal to f on A, such that $\int_B \phi = \int_A^* f$. If f is metrically separable and summable on A_1 and on A_2, it need not be metrically separable on $A_1 + A_2$, but is summable on this set. Jeffery's methods of proving these results are not applicable, but the same results hold, as this paper shows, for functions from the plane to real numbers if Carathéodory outer linear measure and integration with respect to this measure are used.[2]

1. **Equivalence.** Let A be a set with finite outer linear (Carathéodory or Gillespie) measure and $\Gamma[A]$ the set of points of the complement of A where the superior outer density of A is positive. Then, [2], $\overline{A} = A + \Gamma[A]$ is a *massgleiche Hülle* of A, that is, is linearly measurable with linear measure equal to the outer linear measure of A. Thus A is linearly measurable if and only if $\Gamma[A]$ has linear measure zero, that is, $L^*\Gamma[A] = 0$.

[1] Presented to the Society, in part December 30, 1936, under the title *Metric separability and the Hildebrandt integral*, and in part October 28, 1939, under the title *Metric separability*.

[2] The same methods also prove these results if outer Gillespie linear measure (see [3]) is used.

If L^*A_1 and L^*A_2 are finite, then A_1 and A_2 are metrically separable if and only if

$$LA_1\overline{A}_2 = 0 \qquad (\text{or } LA_2\overline{A}_1 = 0).$$

For $L^*(A_1+A_2) \leqq L^*(A_1+\overline{A}_2) \leqq L(\overline{A_1+A_2}) = L^*(A_1+A_2)$ so the equality holds; and then since $\overline{A}_2$ is measurable

$$L^*(A_1 + \overline{A}_2) = L^*A_1 + L\overline{A}_2 - L^*A_1\overline{A}_2 = L^*A_1 + L^*A_2 - L^*A_1\overline{A}_2,$$

so

$$L^*(A_1 + A_2) = L^*A_1 + L^*A_2$$

if and only if $L^*A_1\overline{A}_2 = 0$. In particular if A_1 and A_2 have no points in common they are metrically separable if and only if

$$L^*A_1\Gamma[A_2] = 0 \qquad (\text{or } L^*A_2\Gamma[A_1] = 0).$$

Also if $A = A_1 + \cdots + A_m$, L^*A_i finite, A_i and A_j metrically separable, then $L^*A = L^*A_1 + \cdots + L^*A_m$ and

$$(1) \qquad A + \Gamma[A] = A_1 + \Gamma[A_1] + \cdots + A_m + \Gamma[A_m],$$

both equations following quite simply.

We now prove a theorem for bounded functions on sets of finite outer linear measure that may be extended in the usual way to functions summable, in the sense of outer integrals.

THEOREM. *If A is a plane set with L^*A finite and f is a bounded ($\alpha < f(p) < \beta$, $p \in A$) function metrically separable on A (so $\int_A^* f$ exists) and $B = A + \Gamma[A]$, then there is a function ϕ which is measurable on B, $p \in A$ implies $\phi(p) = f(p)$, and*

$$\int_B \phi(p)dL(B) = \int_A^* f(p)dL^*(A).$$

If $c < d$ are any two numbers, let $A(c, d) = E_p[c \leqq f(p) < d;\ p \in A]$. For each $n = 1, 2, \cdots$ let $\alpha = a_{n,0}, a_{n,1}, \cdots, a_{n,2^n} = \beta$ be a subdivision such that

$$a_{n,i} - a_{n,i-1} = \beta - \alpha/2^n, \qquad i = 1, 2, \cdots, 2^n,$$

and define

$$A_i = A(a_{n,2^n-i};\ a_{n,2^n-i+1}), \qquad i = 1, 2, \cdots, 2^n.$$

The union of these sets is A and A_i and A_j are metrically separable $i \neq j$ (since f is metrically separable on A); so from (1)

$$B = \sum_{i=1}^{2^n} A_i + \Gamma[A_i].$$

We now define for each $n=1, 2, 3, \cdots$

$$\phi_n(p) = \begin{cases} f(p) & \text{if } p \in A, \\ a_{n,2^n-i} & \text{if } p \in \{\Gamma[A_i] - (\Gamma[A(a_{n,2^n-i+1})\beta] + A)\},\ i = 1, 2, \cdots. \end{cases}$$

Thus $\phi_n(p)$ is defined for each point p of B. Since $\phi_n(p) \leqq \phi_{n+1}(p) < \beta$ the limit $\phi(p) = \lim_{n\to\infty} \phi_n(p)$ exists. Clearly $\phi(p) = f(p)$ if $p \in A$.

Also ϕ is a measurable function. First $E_p[\phi_n(p) \geqq a_{n,r}] = \overline{A}(a_{n,r}; \beta) - A(\alpha; a_{n,r})$. However, since f is metrically separable on A, the sets $A(a_{n,r}; \beta)$ and $A(\alpha; a_{n,r})$ are metrically separable and have no points in common

$$L\{\Gamma[A(a_{n,r}; \beta)]A(\alpha; a_{n,r})\} = 0,$$

that is, the measurable set $\overline{A}(a_{n,r}; \beta)$ represents almost all points where $\phi_n(p) \geqq a_{n,r}$. Now if λ is any number $\alpha \leqq \lambda \leqq \beta$, let $a_{n_1,r_1}, a_{n_2,r_2}, \cdots$ where $n_1 \leqq n_2 \leqq \cdots$ be an increasing sequence of points of subdivisions approaching λ. Then the measurable set

$$\prod_{i=1}^{\infty} \overline{A}(a_{n_i,r_i}; \beta)$$

differs from $E_p[\phi(p) \geqq \lambda,\ p \in \beta]$ by a set of measure zero. Thus ϕ is a measurable function.

To prove the integral equality let $\alpha = a_0 < a_1 < \cdots < a_n = \beta$ be a subdivision and define

$$s_i = \underset{p}{E}[p \in A,\ a_i \leqq f(p) < a_{i+1}], \qquad S_i = \underset{p}{E}[p \in B,\ a_i \leqq \phi(p) < a_{i+1}].$$

Then S_i is measurable, $S_i \supset s_i$ and

$$L^*A = L^*s_1 + \cdots + L^*s_m \leqq LS_1 + \cdots + LS_m = LB = L^*A,$$

so $L^*s_i = LS_i$, $i = 1, 2, \cdots, m$, and thus by multiplying by a_i, summing and taking limits we see the desired equality

$$\int_A^* f = \int_B \phi$$

for bounded functions over sets of finite outer linear measure.

2. **Additivity.** If G_1 and G_2, each with finite outer measure, are metrically separable and f is a function metrically separable on each, f is metrically separable on G_1+G_2. However, if G_1 and G_2 are not

metrically separable, then f may not be metrically separable on their union, for example, if G_1 and G_2 are complements on the unit interval each with outer measure unity and $f(x)=1$ if $x \varepsilon G_1$, $f(x)=2$ if $x \varepsilon G_2$. If however, f is metrically separable and the outer integral of f exists on each set G_1 and G_2, then the outer integral exists on G_1+G_2. For let $G_1=G_{11}+G_{12}$ where $G_{12}=G_1(G_2+\Gamma[G_2])$ and $G_{11}=G_1-G_{12}$ and in like manner take $G_2=G_{21}+G_{22}$. Then G_{11} and G_{12} are metrically separable as are G_{21} and G_{22} as well as G_{11} and G_{21}, while no subset of positive outer measure of G_{12} is metrically separable from G_{22} and no subset of G_{22} is metrically separable from G_{12}. We may then apply the following theorem (proved for positive bounded functions, but extendible in the usual way) to obtain the property of additivity over sets of the existence of outer integrals.

THEOREM. *Let A_1 and A_2 be two point sets of the plane with outer linear measure finite such that no part of either with positive outer linear measure is metrically separable from the other. Then if f is a positive and bounded $(0<f(p)<\beta; p \varepsilon A_1+A_2)$ function metrically separable on A_1 and on A_2, the outer integral of f exists on A_1+A_2.*

For each $n=1, 2, 3, \cdots$ let $0=a_{n,0}, a_{n,1}, \cdots, a_{n,2^n}=\beta$ be a subdivision of the interval $(0, \beta)$ into 2^n equal parts. Define

$$s_{n1i} = \underset{p}{E}[p \,\varepsilon\, A_1,\ a_{n,i-1} \leqq f(p) < a_{n,i}],$$

$$s_{n2i} = \underset{p}{E}[p \,\varepsilon\, A_2,\ a_{n,i-1} \leqq f(p) < a_{n,i}], \qquad i = 1, 2, \cdots, 2^n,$$

and

$$e_n = \sum_{i=1}^{2n} \overline{s_{n1i}}\ \overline{s_{n2i}}$$

where $\overline{s_{n1i}}=s_{n1i}+\Gamma[s_{n1i}]$. Then e_n is a measurable subset of the measurable set $\overline{A_1}+\overline{A_2}$. Also let

$$E_n = (\overline{A_1} + \overline{A_2}) - e_n.$$

Since $s_{n+1,1,2i-1}$ and $s_{n+1,1,2i}$ are metrically separable and their union is s_{n1j} we have

$$\bar{s}_{n+1,1,2i-1} + \bar{s}_{n+1,1,2i} = \bar{s}_{n,1,i}$$

and

$$\bar{s}_{n+1,2,2i-1} + \bar{s}_{n+1,2,2i} = \bar{s}_{n,2,i}, \qquad i = 1, 2, \cdots, 2^n.$$

If then $p \,\varepsilon\, e_{n+1}$ there will be a j such that p will be either in

$\bar{s}_{n+1,1,2i-1}\bar{s}_{n+1,2,2i-1}$ or $\bar{s}_{n+1,1,2i}\bar{s}_{n+1,2,2i}$ so in either case will belong to $\bar{s}_{n,1,i}\bar{s}_{n,2,i}$ and so to e_n, that is, $e_{n+1}\subset e_n$. Thus $E_{n+1}\supset E_n$ and the sets

$$e = e_1\cdot e_2 \cdots , \qquad E = E_1 + E_2 + \cdots$$

are clearly metrically separable.

We first prove that the outer integral of f exists over $e(A_1+A_2)$.

Let $b_{ni}=E_p[p \varepsilon e(A_1+A_2),\ a_{n,i-1}\leqq f(p)<a_{n,i}]$, $n=1, 2, 3, \cdots$, $i=1, 2, \cdots, 2^n$. Then $b_{n,i}=e(s_{n,1,i}+s_{n,2,i})$. But $e_n\bar{s}_{n1i}$ and $e\bar{s}_{n2i}$ differ from each other only in a set of measure zero so $e\bar{s}_{n1i}$ and $e\bar{s}_{n2i}$ differ at most by a set of measure zero, so

$$Le(\bar{s}_{n1i} + \bar{s}_{n2i}) = Le\bar{s}_{n1i} = Le\bar{s}_{n2i},$$

so

$$L^*e(s_{n1i} + s_{n2i}) = L^*es_{n1i} = L^*es_{n2i}$$

or

$$L^*b_{ni} = L^*es_{ni} = L^*es_{n2i}.$$

Thus

$$\sum_{i=1}^{2n} a_{ni}L^*b_{ni} = \sum_{i=1}^{2n} a_{ni}L^*es_{n1i} = \sum_{i=1}^{2n} a_{ni}L^*es_{n2i},$$

and consequently the integral of f exists over $e(A_1+A_2)$ and

$$\int^*_{e(A_1+A_2)} f(p)dL(p) = \int^*_{eA_1} f(p)dL(p) = \int^*_{eA_2} f(p)dL(p).$$

We now show that f has an outer integral on $E(A_1+A_2)$ which moreover is the sum of the outer integrals over EA_1 and EA_2.

Let $e_n' =\sum_{i=1}^{2^n} E\bar{s}_{n1i}\bar{s}_{n2i}$ and let $E_n' =E(\overline{A}_1+\overline{A}_2)-e_n'$. Now $e_n' \supset e_{n+1}'$ so $E_n' \subset E_{n+1}'$. Also $\lim_{n\to\infty} L^*e_n' =0$ for otherwise we would have $L^*Ee>0$. Therefore $\lim_{n\to\infty} L^*E_n' =E$ and thus $\lim_{n\to\infty} L^*E(A_1+A_2) =L^*E(A_1+A_2)$, $\lim_{n\to\infty} L^*E_n' A_1=L_1EA_1$ and $\lim_{n\to\infty} L^*E_n' A_2=EA_2$.

Now for $n'>n$ let $h_{n'ji}=E_p[p \varepsilon E_n' A_j,\ a_{n',i-1}\leqq f<a_{n',i}]$, $j=1, 2$; $i=1, 2, \cdots, 2^{n'}$. Then $h_{n'1i}$ and $h_{n'2i}$ are metrically separable, so we have

$$L^*(h_{n'1i} + h_{n'2i}) = L^*h_{n'1i} + L^*h_{n'2i}$$

and thus

$$\int^*_{E_n'(A_1+A_2)} f = \int^*_{E_n'A_1} f + \int^*_{E_n'A_2} f.$$

Also since

$$\int_{E_n'A_1}^{*} f + \int_{(E-E_n')A_1}^{*} f = \int_{EA_1}^{*} f,$$

we have

$$\lim_{n\to\infty} \int_{E_n'A_1}^{*} f = \int_{EA_1}^{*} f, \qquad \lim_{n\to\infty} \int_{E_n'A_2}^{*} f = \int_{EA_2}^{*} f.$$

Then since

$$\int_{E_n'(A_1+A_2)}^{*} f \leqq \int_{E_{n+1}'(A_1+A_2)}^{*} f \leqq \int_{EA_1}^{*} f + \int_{EA_2}^{*} f,$$

we see that

$$\lim_{n\to\infty} \int_{E_n'(A_1+A_2)}^{*} f \text{ exists}$$

and is less than or equal to

$$\int_{EA_1}^{*} f + \int_{EA_2}^{*} f.$$

We may further assert that the equality holds. For $\epsilon > 0$ given there is an N such that $n \geqq N$ implies

$$\int_{E_n'A_j}^{*} f > \int_{EA_j}^{*} f - \frac{\epsilon}{2}, \qquad j = 1, 2,$$

or

$$\int_{E_n'(A_1+A_2)}^{*} f = \int_{E_n'A_1}^{*} f + \int_{E_n'A_2}^{*} f > \int_{EA_1}^{*} f + \int_{EA_2}^{*} f - \epsilon.$$

Thus

$$\int_{A_1+B_1}^{*} f = \int_{E(A_1+A_2)}^{*} f + \int_{e(A_1+A_2)}^{*} f = \int_{EA_1}^{*} f + \int_{EA_2}^{*} f + \int_{e(A_1+A_2)}^{*} f.$$

References

1. R. L. Jeffery, *Relative summability*, Annals of Mathematics, (2), vol. 33 (1932), pp. 443–459.

2. J. F. Randolph, *Some density properties of points sets*, Annals of Mathematics, (2), vol. 37 (1936), pp. 336–344.

3. A. P. Morse and J. F. Randolph, *Gillespie measure*, Duke Mathematical Journal, vol. 6 (1940), pp. 408–419.

Cornell University