R. L. Jeffery [1] investigated an upper integral for functions (from the line to real numbers) not necessarily measurable. Let \( f \) be bounded, \( \alpha < f(x) < \beta \) for \( x \in A \), let \( e_i = E_\varepsilon [a_i \leq f(x) < a_{i+1}] \) where \( \alpha = a_0 < a_1 < \cdots < a_n = \beta \) and consider \( \sum a_i m^* e_i \). If as \( n \) increases and \( \max (a_{i+1} - a_i) \) approaches zero, the limit of this sum exists and is independent of the subdivisions, this limit is the upper integral of \( f \) over \( A \), \( \int_A f(x) \, dx \).

Two point sets with finite outer measure are metrically separable if the outer measure of their sum is the sum of their outer measures. A function is metrically separable on a set \( A \) if for each constant \( \lambda \) the part of \( A \) on which the function is greater than \( \lambda \) is metrically separable from the rest of \( A \). Jeffery proved that metric separability may be made the basis of a comprehensive theory of integration which includes Young, Pierpont, and Lebesgue integration.

All measurable functions are metrically separable, but a function defined over a non-measurable set (and so necessarily non-measurable) may still be metrically separable. However, if \( f \) is metrically separable and possesses an outer integral on a set \( A \), there exists a measurable set \( B \supset A \) and a function \( \phi \), measurable on \( B \) and equal to \( f \) on \( A \), such that \( \int_B \phi = \int_A f \). If \( f \) is metrically separable and summable on \( A_1 \) and on \( A_2 \), it need not be metrically separable on \( A_1 + A_2 \), but is summable on this set. Jeffery's methods of proving these results are not applicable, but the same results hold, as this paper shows, for functions from the plane to real numbers if Carathéodory outer linear measure and integration with respect to this measure are used.\(^2\)

1. Equivalence. Let \( A \) be a set with finite outer linear (Carathéodory or Gillespie) measure and \( \Gamma [A] \) the set of points of the complement of \( A \) where the superior outer density of \( A \) is positive. Then, \( [2] \), \( \mathcal{A} = A + \Gamma [A] \) is a massgleiche Hülle of \( A \), that is, is linearly measurable with linear measure equal to the outer linear measure of \( A \). Thus \( A \) is linearly measurable if and only if \( \Gamma [A] \) has linear measure zero, that is, \( L^* \Gamma [A] = 0 \).

\(^1\) Presented to the Society, in part December 30, 1936, under the title Metric separability and the Hildebrandt integral, and in part October 28, 1939, under the title Metric separability.

\(^2\) The same methods also prove these results if outer Gillespie linear measure (see \([3]\)) is used.
If \( L^*A_1 \) and \( L^*A_2 \) are finite, then \( A_1 \) and \( A_2 \) are metrically separable if and only if

\[
L A_1 A_2 = 0 \quad \text{(or } L A_2 A_1 = 0). \]

For \( L^*(A_1 + A_2) \leq L^*(A_1 + A_2) \leq L(A_1 + A_2) = L^*(A_1 + A_2) \), so the equality holds, and then since \( A_2 \) is measurable

\[
L^*(A_1 + A_2) = L^*A_1 + L A_2 - L^*A_1 A_2 = L^*A_1 + L^*A_2 - L^*A_1 A_2,
\]

so

\[
L^*(A_1 + A_2) = L^*A_1 + L^*A_2
\]

if and only if \( L^*A_1 A_2 = 0 \). In particular if \( A_1 \) and \( A_2 \) have no points in common they are metrically separable if and only if

\[
L^*A_1 \Gamma [A_2] = 0 \quad \text{(or } L^*A_2 \Gamma [A_1] = 0). \]

Also if \( A = A_1 + \cdots + A_m, \ L^*A_i \) finite, \( A_i \) and \( A_j \) metrically separable, then \( L^*A = L^*A_1 + \cdots + L^*A_m \) and

(1)

\[
A + \Gamma [A] = A_1 + \Gamma [A_1] + \cdots + A_m + \Gamma [A_m],
\]

both equations following quite simply.

We now prove a theorem for bounded functions on sets of finite outer linear measure that may be extended in the usual way to functions summable, in the sense of outer integrals.

**Theorem.** If \( A \) is a plane set with \( L^*A \) finite and \( f \) is a bounded \((a < f(p) < b, \ p \in A)\) function metrically separable on \( A \) (so \( f^*f \) exists) and \( B = A + \Gamma [A] \), then there is a function \( \phi \) which is measurable on \( B \), \( p \in A \) implies \( \phi(p) = f(p) \), and

\[
\int_B \phi(p)dL(B) = \int_A f(p)dL^*(A).
\]

If \( c < d \) are any two numbers, let \( A(c, d) = E_p [c \leq f(p) < d; \ p \in A] \). For each \( n = 1, 2, \cdots \) let \( \alpha = a_{n,0}, a_{n,1}, \cdots, a_{n,2^n} = \beta \) be a subdivision such that

\[
a_{n,i} - a_{n,i-1} = \beta - \alpha/2^n, \quad i = 1, 2, \cdots, 2^n,
\]

and define

\[
A_i = A(a_{n,2^n-i}; a_{n,2^n-i+1}), \quad i = 1, 2, \cdots, 2^n.
\]

The union of these sets is \( A \) and \( A_i \) and \( A_j \) are metrically separable \( i \neq j \) (since \( f \) is metrically separable on \( A \)); so from (1)
We now define for each $n = 1, 2, 3, \ldots$

$$\phi_n(p) = \begin{cases} f(p) & \text{if } p \in A, \\ a_n, a_{n-i} & \text{if } p \in \{ \Gamma[A_i] \cap (\Gamma[A(a_n, a_{n-i+1}) - A] + A) \}, i = 1, 2, \ldots. \end{cases}$$

Thus $\phi_n(p)$ is defined for each point $p$ of $B$. Since $\phi_n(p) \leq \phi_{n+1}(p) < \beta$ the limit $\phi(p) = \lim_{n \to \infty} \phi_n(p)$ exists. Clearly $\phi(p) = f(p)$ if $p \in A$.

Also $\phi$ is a measurable function. First $E_p[\phi_n(p) \geq a_{n,r}] = A(a_n, r; \beta) - A(\alpha; a_n, r)$. However, since $f$ is metrically separable on $A$, the sets $A(a_n, r; \beta)$ and $A(\alpha; a_n, r)$ are metrically separable and have no points in common

$$L\{ \Gamma[A(a_n, r; \beta)]A(\alpha; a_n, r) \} = 0,$$

that is, the measurable set $A(a_n, r; \beta)$ represents almost all points where $\phi_n(p) \geq a_{n,r}$. Now if $\lambda$ is any number $\alpha \leq \lambda \leq \beta$, let $a_{n_1, r_1}, a_{n_2, r_2}, \ldots$ where $n_1 \leq n_2 \leq \cdots$ be an increasing sequence of points of subdivisions approaching $\lambda$. Then the measurable set

$$\prod_{i=1}^\infty A(a_{n_i, r_i}; \beta)$$

differs from $E_p[\phi(p) \geq \lambda, p \in \beta]$ by a set of measure zero. Thus $\phi$ is a measurable function.

To prove the integral equality let $\alpha = a_0 < a_1 < \cdots < a_n = \beta$ be a subdivision and define

$$s_i = E[p \in A, a_i \leq f(p) < a_{i+1}], \quad S_i = E[p \in B, a_i \leq \phi(p) < a_{i+1}].$$

Then $S_i$ is measurable, $S_i \supseteq s_i$ and

$$L^*A = L^*s_1 + \cdots + L^*s_m \leq Ls_1 + \cdots + Ls_m = LB = L^*A,$$

so $L^*s_i = LS_i$, $i = 1, 2, \ldots, m$, and thus by multiplying by $a_i$, summing and taking limits we see the desired equality

$$\int_A f = \int_B \phi$$

for bounded functions over sets of finite outer linear measure.

2. Additivity. If $G_1$ and $G_2$, each with finite outer measure, are metrically separable and $f$ is a function metrically separable on each, $f$ is metrically separable on $G_1 + G_2$. However, if $G_1$ and $G_2$ are not
metrically separable, then $f$ may not be metrically separable on their union, for example, if $G_1$ and $G_2$ are complements on the unit interval each with outer measure unity and $f(x) = 1$ if $x \in G_1$, $f(x) = 2$ if $x \in G_2$. If however, $f$ is metrically separable and the outer integral of $f$ exists on each set $G_1$ and $G_2$, then the outer integral exists on $G_1 + G_2$. For let $G_i = G_{i1} + G_{i2}$ where $G_{i2} = G_i(G_2 + \Gamma [G_i])$ and $G_{i1} = G_i - G_{i2}$ and in like manner take $G_2 = G_{12} + G_{22}$. Then $G_{i1}$ and $G_{i2}$ are metrically separable as are $G_{21}$ and $G_{22}$ as well as $G_{11}$ and $G_{21}$, while no subset of positive outer measure of $G_{12}$ is metrically separable from $G_{22}$ and no subset of $G_{22}$ is metrically separable from $G_{12}$. We may then apply the following theorem (proved for positive bounded functions, but extendible in the usual way) to obtain the property of additivity over sets of the existence of outer integrals.

**Theorem.** Let $A_1$ and $A_2$ be two point sets of the plane with outer linear measure finite such that no part of either with positive outer linear measure is metrically separable from the other. Then if $f$ is a positive and bounded ($0 < f(p) < \beta; p \in A_1 + A_2$) function metrically separable on $A_1$ and on $A_2$, the outer integral of $f$ exists on $A_1 + A_2$.

For each $n = 1, 2, 3, \ldots$ let $0 = a_{n,0}, a_{n,1}, \ldots, a_{n,2^n} = \beta$ be a subdivision of the interval $(0, \beta)$ into $2^n$ equal parts. Define

$$s_{n1i} = E[p \in A_1, a_{n,i-1} \leq f(p) < a_{n,i}],$$

$$s_{n2i} = E[p \in A_2, a_{n,i-1} \leq f(p) < a_{n,i}], \quad i = 1, 2, \ldots, 2^n,$$

and

$$e_n = \sum_{i=1}^{2^n} s_{n1i} + s_{n2i},$$

where $s_{n1i} + s_{n1i} + s_{n11}$. Then $e_n$ is a measurable subset of the measurable set $A_1 + A_2$. Also let

$$E_n = (A_1 + A_2) - e_n.$$

Since $s_{n+1,1,2i-1}$ and $s_{n+1,1,2i}$ are metrically separable and their union is $s_{n1i}$ we have

$$\bar{s}_{n+1,1,2i-1} + \bar{s}_{n+1,1,2i} = \bar{s}_{n,1,i}$$

and

$$\bar{s}_{n+1,2,2i-1} + \bar{s}_{n+1,2,2i} = \bar{s}_{n,2,i}, \quad i = 1, 2, \ldots, 2^n.$$

If then $p \in e_{n+1}$ there will be a $j$ such that $p$ will be either in
\[ s_{n,1,1,2i-1}s_{n,1,2,2i-1} \text{ or } s_{n,1,1,2i}s_{n,1,2,2i} \] so in either case will belong to 
\[ s_{n,1,i}s_{n,1,2,i} \] and so to \( e_n \), that is, \( e_{n+1} \sqsubset e_n \). Thus \( E_{n+1} \supset E_n \) and the sets 
\[ e = e_1 \cdot e_2 \cdots, \quad E = E_1 + E_2 + \cdots \]
are clearly metrically separable.

We first prove that the outer integral of \( f \) exists over \( e(A_1 + A_2) \).

Let \( b_n = E_p[p \in e(A_1 + A_2), a_{n,i-1} \leq f(p) < a_{n,i}], n = 1, 2, 3, \ldots, i = 1, 2, \ldots, 2^n \). Then \( b_n = e(s_{n,1,i} + s_{n,2,i}) \). But \( e_n s_{n,1} \) and \( e \) \( s_{n,2} \) differ from each other only in a set of measure zero so \( e s_{n,1} \) and \( e s_{n,2} \) differ at most by a set of measure zero, so

\[ L e(s_{n,1} + s_{n,2}) = L e s_{n,1} = L e s_{n,2}, \]

so

\[ L^* e(s_{n,1} + s_{n,2}) = L^* e s_{n,1} = L^* e s_{n,2} \]
or

\[ L^* b_n = L^* e s_{n,1} = L^* e s_{n,2}. \]

Thus

\[ \sum_{t=1}^{2^n} a_n L^* b_n = \sum_{t=1}^{2^n} a_n L^* e s_{n,1} = \sum_{t=1}^{2^n} a_n L^* e s_{n,2}, \]

and consequently the integral of \( f \) exists over \( e(A_1 + A_2) \) and

\[ \int_{e(A_1 + A_2)} f(p) dL(p) = \int_{eA_1} f(p) dL(p) + \int_{eA_2} f(p) dL(p). \]

We now show that \( f \) has an outer integral on \( E(A_1 + A_2) \) which moreover is the sum of the outer integrals over \( E A_1 \) and \( E A_2 \).

Let \( e_n' = \sum_{t=1}^{2^n} E s_{n,1}s_{n,2} \) and let \( E_n' = E(E_1 + A_2) - e_n' \). Now \( e_n' \supset e_{n+1}' \) so \( E_n' \subset E_{n+1}' \). Also \( \lim_{n \to \infty} L^* e_{n}' = 0 \) for otherwise we would have \( L^* E_1 > 0 \). Therefore \( \lim_{n \to \infty} L^* E_n' = E \) and thus \( \lim_{n \to \infty} L^* E_n A_1 = L^* E_1 A_1 \) and \( \lim_{n \to \infty} L^* E_n A_2 = E A_2 \).

Now for \( n' > n \) let \( h_{n',i} = E_p[p \in E_n A_i, a_{n',i-1} \leq f < a_{n',i}], j = 1, 2; i = 1, 2, \ldots, 2^{n'} \). Then \( h_{n,1}' \) and \( h_{n,2}' \) are metrically separable, so we have

\[ L^*(h_{n,1}' + h_{n,2}') = L^* h_{n,1}' + L^* h_{n,2}' \]

and thus

\[ \int_{E_n'(A_1 + A_2)} f = \int_{E_n'A_1} f + \int_{E_n'A_2} f. \]
Also since
\[ \int_{E 'A_1} f + \int_{(E-E_1)A_1} f = \int_{EA_1} f, \]
we have
\[ \lim_{n \to \infty} \int_{E n 'A_1} f = \int_{EA_1} f, \quad \lim_{n \to \infty} \int_{E n 'A_2} f = \int_{EA_2} f. \]

Then since
\[ \int_{E n ' (A_1+A_2)} f \leq \int_{E n '+1 (A_1+A_2)} f \leq \int_{EA_1} f + \int_{EA_2} f, \]
we see that
\[ \lim_{n \to \infty} \int_{E n ' (A_1+A_2)} f \]
exists and is less than or equal to
\[ \int_{EA_1} f + \int_{EA_2} f. \]

We may further assert that the equality holds. For \( \epsilon > 0 \) given there is an \( N \) such that \( n \geq N \) implies
\[ \int_{E n 'A_j} f > \int_{EA_j} f - \frac{\epsilon}{2}, \quad j = 1, 2, \]
or
\[ \int_{E n ' (A_1+A_2)} f = \int_{E n 'A_1} f + \int_{E n 'A_2} f > \int_{EA_1} f + \int_{EA_2} f - \epsilon. \]

Thus
\[ \int_{A_1+B_1} f = \int_{E(A_1+A_2)} f + \int_{e(A_1+A_2)} f = \int_{EA_1} f + \int_{EA_2} f + \int_{e(A_1+A_2)} f. \]

REFERENCES

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