A PROPERTY OF A SIMPLY ORDERED SET[1]

K. W. FOLLEY

Sierpinski[2] has shown that the set of real numbers of the interval (0, 1) may be decomposed into c disjoint subsets, each of power less than c, and such that the sum of every c of these subsets has at least one point in common with every perfect subset of the interval.

The object of the present paper is to show that the same method of proof may be used to prove an analogous theorem concerning a more general type of set.

DEFINITIONS. *A simply ordered set M is a set such that if any two of its elements are given it is known which one precedes.*

A subset of M is said to be cofinal (coinitial) with M if no element of M follows (precedes) all the elements of the subset.

An η_α subset of M is one which is neither cofinal nor coinitial with any subset of M of power less than $\aleph_\alpha$ and which contains no pair of neighboring subsets both of which have power less than $\aleph_\alpha$.

Various properties of simply ordered sets M containing everywhere dense η_α subsets, including a discussion of the perfect subsets of M, were discussed by the writer in a previous paper.[3]

THEOREM 1. *Let M be a simply ordered set containing an everywhere dense η_α subset N. There exists a decomposition of M into $2^{\aleph_\alpha}$ disjoint subsets, each of power less than $2^{\aleph_\alpha}$, and such that the sum of every $2^{\aleph_\alpha}$ of these subsets has at least one point in common with every perfect subset of M.*

PROOF. Let ϕ be the smallest ordinal number of power $2^{\aleph_\alpha}$. A transfinite sequence of type ϕ formed of all the points of M exists, namely,

$$m_1, m_2, m_3, \cdots, m_\xi, \cdots, \qquad \xi < \phi. \tag{1}$$

The perfect subsets of M having power $2^{\aleph_\alpha}$ may be arranged in the form of a transfinite sequence of type ϕ as follows:

$$M_1, M_2, M_3, \cdots, M_\xi, \cdots, \qquad \xi < \phi. \tag{2}$$

Let us now define a transfinite sequence $\{H_\xi\}_{\xi<\phi}$ of subsets of M: H_1 is formed of the single element m_1.

[1] Presented to the Society, November 22, 1940.

[2] Fundamenta Mathematicae, vol. 24 (1935), pp. 8–11.

[3] Proceedings of the Royal Society of Canada, Section III, vol. 22 (1928), pp. 225–239.

Now, let β be any ordinal number between 1 and ϕ. Suppose we have defined all sets H_ξ, where $\xi<\beta$, and suppose further that the power of H_ξ is not greater than $\bar{\xi}$. Let $S_\beta=\sum_{\xi<\beta}H_\xi$. Then S_β is evidently of power not greater than $\bar{\beta}^2<2^{\aleph_\alpha}$.

We shall define a transfinite sequence $\{p_\xi^\beta\}$ as follows. M_1 being of power $2^{\aleph_\alpha}$, and S_β being of power less than $2^{\aleph_\alpha}$, M_1-S_β contains points. Let p_1^β be the first element of (1) which belongs to M_1-S_β. Now let η be any ordinal number between 1 and β, and let us suppose we have already defined the points p_ξ^β, where $\xi<\eta$. Let all these points form the set T_η^β of power not greater than $\bar{\eta}\leqq\bar{\beta}<2^{\aleph_\alpha}$. The set $M_\eta-(S_\beta+T_\eta)$ contains points. Let p_η^β be the first point of (1) which belongs to the set $E_\eta-(S_\beta+T_\eta)$.

Define H_β as the set of all points p_ξ^β, where $\xi<\beta$; it is a set of power not greater than $\bar{\beta}<2^{\aleph_\alpha}$.

The sets H_ξ, with $\xi<\phi$, are thus defined by transfinite induction. They are evidently disjoint subsets of M, each of power less than $2^{\aleph_\alpha}$. The set $R=M-\sum_{\xi<\phi}H_\xi$ is of power not greater than $2^{\aleph_\alpha}$. Hence the elements of R may be arranged in the form of a transfinite sequence of type $\psi\leqq\phi$ as follows:

$$q_1, q_2, q_3, \cdots, q_\xi, \cdots, \qquad \xi<\psi.$$

Let $N_\xi=H_\xi+q_\xi$ for $\xi<\psi$, and if $\psi<\phi$, $M_\xi=H_\xi$ for $\psi\leqq\xi<\phi$.

The sets N_ξ are disjoint subsets of M, each of power less than $2^{\aleph_\alpha}$. Moreover $M=\sum_{\xi<\phi}N_\xi$.

Now, let F be the sum of any $2^{\aleph_\alpha}$ of the sets N_ξ, $\xi<\phi$. If μ is any ordinal number less than ϕ, there is an ordinal number β such that $\mu<\beta<\phi$, and such that N_β belongs to F. Since $\beta>\mu$ we have p_μ^β belonging to H_μ and therefore to N_μ. Besides p_μ^β belongs to M_μ. Hence p_μ^β is a point of $M_\mu\cdot H_\beta$ and so of $F\cdot M_\mu$.

It follows that F has at least one point in common with every subset M_ξ of (2), which was to be proved.

THEOREM 2. *The generalized hypothesis of the continuum* ($2^{\aleph_\alpha}=\aleph_{\alpha+1}$) *is equivalent to the following statement:*

The set M may be decomposed into disjoint subsets, each of power not greater than $\aleph_\alpha$, *such that the sum of any class of more than* $\aleph_\alpha$ *of them has at least one point in common with every perfect subset of M.*

PROOF. (a) If the generalized hypothesis of the continuum is assumed, Theorem 1 becomes the second part of Theorem 2.

(b) Suppose that the second part of Theorem 2 is true. There are $2^{\aleph_\alpha}$ perfect subsets of[4] M. Each disjoint subset of M having a power

[4] See K. W. Folley, loc. cit., p. 232.

not greater than $\aleph_\alpha$, it follows that a class of $\aleph_{\alpha+1}$ of them has power $\aleph_{\alpha+1}$. Such a class has at least one point in common with every perfect subset of M. Thus, there results the inequality

$$\aleph_{\alpha+1} \leqq 2^{\aleph_\alpha}.$$

The inequality in the opposite sense being well known, the generalized hypothesis of the continuum follows.

WAYNE UNIVERSITY

UNIQUENESS THEOREMS FOR RATIONAL FUNCTIONS[1]

FRITZ HERZOG

In his book on the theory of meromorphic functions,[2] R. Nevanlinna proved a number of "uniqueness theorems." The most important of them[3] states that if two functions $w=f(x)$ and $w=g(x)$, meromorphic in the whole x-plane, assume *five* values of w (finite or infinite) at the same points x they must be identical. If we understand by the *distribution* of a function $w=\phi(x)$ with respect to a given value of w simply the set of all points x where $\phi(x)$ assumes that value w, *regardless of multiplicity*, we may state the above theorem in the following way: Two meromorphic functions which have identical distributions with respect to five values of the dependent variable must be identical. In proving this theorem, Nevanlinna explicitly assumes the functions to be transcendental (i.e., not rational).[3] It is trivial, however, that the theorem would apply to two rational functions $w=f(x)$ and $w=g(x)$ as well, which can be easily seen by considering the transcendental functions $w=f(e^x)$ and $w=g(e^x)$.

The example of the functions $w=e^x$ and $w=e^{-x}$, which have identical distributions with respect to the four values $w=1, -1, 0, \infty$, shows that five is the smallest number for which the above-mentioned uniqueness theorem holds true. It will be shown in this paper that such is not the case for rational functions for which five may, indeed, be replaced by *four*. (See Theorem I.)

The question arises as to what can be said about two rational functions that have identical distributions with respect to only *three*

[1] Presented to the Society, February 24, 1940.

[2] Rolf Nevanlinna, *Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes*, Paris, 1929.

[3] See loc. cit., p. 109.