The max $|l_1^{(n)}(x)|$ is attained at $x = \pm 1$ since (1) $\theta_{k+1} - \theta_k \leq 2\pi/(2n + \alpha + \beta - 1)$ provided $\frac{1}{2} \leq \alpha, \beta \leq \frac{3}{2}$ and $x_k = \cos \theta_k$. Using the second asymptotic formula and the fact that $n\theta_k \rightarrow j_k$ as $n \rightarrow \infty$ where $j_k$ is the $k$th positive zero of $J_{\beta-1}(x)$, we find that

$$|l_1^{(n)}(1)| \rightarrow (\frac{1}{2} j_k)^{\beta - 2} |\Gamma(\beta)J_{\beta}(j_k)|^{-1}$$

as $n \rightarrow \infty$, $k$ constant, $l_1^{(n)}(-1) \rightarrow 0$ which proves the theorem:

**Theorem 7.** Max $|l_1^{(n)}(x)| \rightarrow (\frac{1}{2} j_1)^{\beta - 2} |\Gamma(\beta)J_{\beta}(j_1)|^{-1}$ as $n \rightarrow \infty$ (where $\frac{1}{2} \leq \alpha, \beta \leq \frac{3}{2}$, $j_1$ is first positive zero of $J_{\beta-1}(x)$).

A similar result holds for $l_1^{(n)}(x)$ if $\beta$ is replaced by $\alpha$.

For Legendre polynomials ($\alpha = \beta = 1$) this limit is approximately 1.602. For $\alpha = \beta = \frac{1}{2}$ and $\alpha = \beta = \frac{3}{2}$ the limit of Theorem 7 is also an upper bound for max $|l_1^{(n)}(x)|$ and max $|l_1^{(n)}(x)|$. Whether this is true, in general, remains unanswered.

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**AN INVARIANCE THEOREM FOR SUBSETS OF $S^n$**

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The purpose of this paper is to establish the following.

**Invariance Theorem.** Let $A$ and $B$ be two homeomorphic subsets of the $n$-sphere $S^n$. If the number of components of $S^n - A$ is finite, then it is equal to the number of components of $S^n - B$.

In the case when $A$ and $B$ are closed this theorem is a very well known consequence of Alexander's duality theorem and its generalizations. In our case we also derive our result as a consequence of a duality theorem. However, the duality is established only for the dimension $n-1$.

Given a metric space $X$ we shall say that $\Gamma^k$ is a $k$-cycle in $X$ if there is a compact subset $A$ of $X$ such that $\Gamma^k$ is a $k$-dimensional convergent (Vietoris) cycle in $A$ with coefficients modulo 2. We shall write $\Gamma^k \sim 0$ if $\Gamma^k \sim 0$ holds in some compact subset of $X$. The homology group of $X$ obtained this way will be denoted by $\mathfrak{H}^k(X)$; the corresponding connectivity number, by $\rho^k(X)$. The number $\rho^k(X)$ can be either finite or $\infty$.

1 Presented to the Society, December 28, 1939.
Duality theorem. Let $A \subset S^n$ and let $z_0, z_1, \ldots, z_m$ belong to $m + 1$ different quasi-components$^3$ of $S^n - A$. There are $m$ linearly independent (modulo 2) $(n-1)$-cycles

(1) \[ \Gamma_1^{n-1}, \ldots, \Gamma_m^{n-1} \]

of $A$ such that

(2) \[ v(\Gamma_i^{n-1}, \gamma_j^0) = \delta_{ij}, \quad i, j = 1, \ldots, m, \]

where $\gamma_j^0$ is the 0-cycle $z_0 + z_i$ (consisting of the two points $z_0, z_i$ each of them with coefficient 1) and $v(\Gamma, \gamma)$ is the linking number.

In case $S^n - A$ has only $m + 1$ quasi-components, the cycles (1) form a basis for $3C^{n-1}(A)$.

Proof. In case $A$ is closed the theorem turns out to be a particular case of the generalized Alexander duality theorem.$^3$ We shall prove our theorem for arbitrary sets $A$ using the theorem for closed sets.

Since $z_0, z_1, \ldots, z_m$ belong to $m + 1$ different quasi-components of $S^n - A$ there is a decomposition $S^n - A = A_0 + A_1 + \cdots + A_m$ such that $z_i \subseteq A_i$ and $A_i A_j + A_j A_i = 0$ for $i \neq j$, $i, j = 0, 1, \ldots, m$. Let $B_0, B_1, \ldots, B_m$ be open disjoint sets such that $A_i \subseteq B_i$ for $i = 0, 1, \ldots, m$ and let $B = S^n - (B_0 + B_1 + \cdots + B_m)$. Clearly $B$ is a closed subset of $A$ and $z_0, z_1, \ldots, z_m$ belong to $m + 1$ different quasi-components (equals components) of $S^n - B$.

Applying the duality theorem to the closed set $B$ we obtain the cycles (1) satisfying (2). In order to prove that they determine linearly independent elements modulo 2 of $3C^{n-1}(A)$ consider a cycle $\Gamma^{n-1} = a_1 \Gamma_1^{n-1} + \cdots + a_m \Gamma_m^{n-1}$ where $a_i = 0, 1$. It follows from (2) that $v(\Gamma^{n-1}, \gamma_j^0) = a_j$. Therefore $\Gamma^{n-1} \sim 0$ in $A$ implies $a_1 = \cdots = a_m = 0$.

Suppose now that $S^n - A$ consists of exactly $m + 1$ quasi-components. It follows that the sets $A_0, A_1, \ldots, A_m$ are connected.

Let $\Gamma^{n-1}$ be an $(n-1)$-cycle of $A$ contained in some closed set $D \subset A$. Let $E_i$ be the component of $S^n - (B + D)$ containing $A_i$ ($i = 0, 1, \ldots, m$) and let $E = S^n - (E_0 + E_1 + \cdots + E_m)$. It follows that (1°) $E$ is a closed subset of $A$, (2°) $S^n - E$ consists of exactly $m + 1$ quasi-components (equals components), (3°) the points

$^3$ Two points $x_i, x_2 \in X$ belong to the same quasi-component of $X$ if there is no decomposition $X = A_1 + A_2$ such that $x_1 \in A_1, x_2 \in A_2$ and $A_1 A_2 + A_2 A_1 = 0$. If the number of quasi-components of $X$ is finite then every quasi-component is a component.

$z_0, z_1, \ldots, z_m$ belong to different quasi-components of $S^n - E$, 
(4°) the cycles (1) and $\Gamma^{n-1}$ are contained in $E$. According to the 
duality theorem for closed sets the cycles (1) form a basis for $\mathcal{C}^{n-1}(E)$. 
This implies the existence of $a_1, a_2, \cdots, a_m (a_i = 0, 1)$ such that
\[
\Gamma^{n-1} \sim a_1 \Gamma_1^{n-1} + \cdots + a_m \Gamma_m^{n-1} \text{ in } E.
\]
This proves the theorem since $E \subseteq A$.

Given a metric space $X$ let the number $b_0(X)$ be defined as follows:
- $b_0(X) = 0$ if $X = 0$,
- $b_0(X) = m$ if $X \neq 0$ and $X$ has exactly $m + 1$ components,
- $b_0(X) = \infty$ if $X$ has an infinity of components.

Clearly the value of $b_0(X)$ remains unchanged if we replace in its 
definition components by quasi-components. The duality theorem im­
plies therefore the following:

(I) For every subset $A$ of $S^n$ we have

$$p^{n-1}(A) = b_0(S^n - A).$$

(II) For every two homeomorphic subsets $A$ and $B$ of $S^n$ we have

$$b_0(S^n - A) = b_0(S^n - B).$$

The invariance theorem stated in the introduction follows directly 
from (II).

If $X$ consists of an infinity of components, then instead of taking $b_0(X) = \infty$ we could define $b_0(X)$ to be the cardinal number cor­
responding to the class of all components of $X$. Similarly $p^k(X)$ could 
be redefined as a cardinal number. But with these new definitions (I) 
and (II) are no longer true. In fact, let $A$ be a subset of $S^1$ such that $S^1 - A$ is closed and enumerably infinite, and let $B$ be a subset of $S^1$ such that $S^1 - B$ is perfect and non-dense. It is clear that $A$ and $B$ 
are homeomorphic, that $b_0(S^1 - A) = p^0(A) = p^0(B) = \aleph_0$, and that $b_0(S^1 - B) = 2^\aleph_0$.

That (II) is no longer true was first pointed out to me by Dr. L. Zippin.