THE GENERALIZATION OF A LEMMA OF M. S. KAKEYA

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We shall prove the following:

**Lemma.** It is always possible to find the unique polynomial

\[ \phi^*(z) = \sum_{k=0}^{2s} \gamma_k z^k \]

of degree \( 2s \) possessing the following properties:

I. \( \phi^*(z) = ci^2(z)\tau(z)\tau^*(z), \quad c = \text{const.,} \)

the polynomial \( i(z) \) of degree \( \sigma \leq s \) having all roots in the domain \( |z| > 1 \):

\[ i(z) = \prod_{i=1}^{\sigma} (z - a_i), \quad |a_i| > 1, \quad i = 1, 2, \ldots, \sigma, \]

and the polynomial \( \tau(z) \) being of degree \( \nu = s - \sigma \):

\[ \tau(z) = \prod_{i=1}^{\nu} (z - \alpha_i), \quad \tau^*(z) = z^\nu \left( \frac{1}{z} \right) = \prod_{i=1}^{\nu} (1 - z\alpha_i). \]

II. It is subject to the conditions

\[ \omega_i(\phi^*) = \sum_{k=0}^{2s} \gamma_k c^{(i)}_k = d_i, \quad i = 0, 1, \ldots, s, \]

the given linear functionals \( \omega_i \) being such that every polynomial \( \phi(z) \) of degree \( n \geq 2s \) for which

\[ \omega_i(\phi) = \sum_{k=0}^{2s} \gamma_k c^{(i)}_k = 0, \quad (i = 0, 1, \ldots, s), \quad \phi(z) = \sum_{k=0}^{n} \gamma_k z^k, \]

has \( s + 1 \) roots at least in the domain \( |z| < 1 \).

In the particular case when

\[ \omega_i(\phi) = \phi^{(i)}(z_k), \quad |z_k| < 1, \]

this lemma has been proved by M. S. Kakeya \[1\];\(^1\)

without being aware of his result we have proved this lemma in the case\(^2\)

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\(^1\) Numbers in brackets refer to the bibliography at the end.

\(^2\) In \[1\] and \[2\] one may find the application of this lemma to some extremal problems.

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\[ \omega_i(\phi) = \frac{1}{i!} \left( \frac{d^i \phi}{dz^i} \right)_{z=0}, \quad i = 0, 1, \ldots, s. \]

In order to prove this lemma in the most general case we consider the following extremal problem:

**PROBLEM. To find the minimum of the integral**

\[ L(b) = \int_0^{2\pi} |t(z)|^2 b(\theta) d\theta, \quad z = e^{i\theta}, \]

\( t(z) \) being the given polynomial of degree \( s \) with \( t(0) \neq 0 \) and \( b(\theta) \) being a trigonometric polynomial of order \( n \geq 2s \):

\[ b(\theta) = R \left\{ z^n \phi \left( \frac{1}{z} \right) \right\} = R \sum_{k=0}^{n} \gamma_k \phi^{(n-k)} \theta, \quad z = e^{i\theta}, \]

*subject to the conditions*\(^8\)

\[ \omega_i(b) = \omega_i(\phi) = \sum_{k=0}^{2s} \gamma_k c_k^{(i)} = d_i, \quad i = 0, 1, \ldots, s. \]

The fundamental property of our functionals \( \omega_i \) yields at once that every trigonometric polynomial \( b(\theta) \) subject to the conditions

\[ \omega_i(b) = 0, \quad i = 0, 1, \ldots, s, \]

has in \((0, 2\pi)\) no more than \( 2(n-s-1) \) changes of sign. It is clear that there exists a solution of our problem. Further, the necessary conditions for an extremum are

\[ \text{sgn} \ b^*(\theta) \ |t(z)|^2 = R \sum_{k=n-2s}^{\infty} A_k z^k, \quad z = e^{i\theta}, \]

whence we find at once that the Fourier expansion of \( \text{sgn} \ b^*(\theta) \) is of the form

\[ \text{sgn} \ b^*(\theta) = R \sum_{k=n-s}^{\infty} B_k z^k, \quad z = e^{i\theta}. \]

We have shown in \([2]\) that every trigonometric polynomial with this property must be of the form

\[ b^*(\theta) = R \left\{ \hat{c} z^{n-2s+\sigma} q(z) \right\} \tau(z) \tau(1/z), \quad z = e^{i\theta}, \]

\( q(z) \) being a polynomial of degree \( \sigma \leq s \) all of whose roots lie in the domain \( |z| < 1 \), and \( \tau(z) \) being a polynomial of degree \( \nu = s - \sigma \).

\(^8\) The functionals \( \omega_i \) are the same as above.
The polynomial $b^*(\theta)$ for which the minimum is attained is unique. If there were two such polynomials, $b^*_1(\theta)$ and $b^*_2(\theta)$, then we would have

$$L(b^*_1) \leq L\left(\frac{b^*_1 + b^*_2}{2}\right) \leq L(b^*_2);$$

then $b^*_1(\theta)$ and $b^*_2(\theta)$ would change sign at the same points, that is, the polynomial

$$b^*_1(\theta) - b^*_2(\theta) = \mathcal{R}\left\{z^{n-2s+r}q^2(z)\right\}\left\{c_1 | \tau_1(z)|^2 - c_2 | \tau_2(z)|^2\right\}, \quad z = e^{i\theta},$$

would have at least $2(n-v)$ changes of sign in $(0, 2\pi)$; but since

$$\omega_i(b^*_1 - b^*_2) = 0, \quad i = 0, 1, \ldots, s,$

the polynomial $b^*_1(\theta) - b^*_2(\theta)$ cannot have more than $2(n-s-1)$ changes of sign in $(0, 2\pi)$; this contradiction proves the unicity of the polynomial solving our problem. Thus we find that there exists the unique polynomial $b^*(\theta)$ minimizing (1) under conditions (2) and it must be of the form

$$b^*(\theta) = \mathcal{R}\left\{z^{n-2s+r}q^2(z)\tau(z)\tau(1/z)\right\} = \mathcal{R}\left\{z^n + z^{n-1} + \cdots + z^{2s}q^{n-2s}\right\}, \quad z = e^{i\theta}.$$

Since the real parts of two polynomials coincide on the unit circle, these polynomials are identical, that is,

$$\bar{c}z^{n-2s}q^2(z)\tau(z)\tau^*(z) = z^n + z^{n-1} + \cdots + z^{2s},$$

whence we find finally

$$f^*(z) = q^* + z^* + \cdots + z^{2s} = ci^2(z)\tau(z)\tau^*(z),$$

where

$$i(z) = q^*(z) = z^*q(1/z).$$

Thus we have found the polynomial $f^*(z)$ satisfying all the conditions of our lemma.

**Bibliography**
