PROOF OF A THEOREM OF HALL

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In the Journal of the London Mathematical Society for July, 1937, Mr. Philip Hall gave a proof of the theorem, "If a group \( G \) of order \( g \) has a subgroup of order \( m \) for every divisor \( m \) of \( g \) such that \( (m, g/m) = 1 \), then \( G \) is a soluble group." The proof is a very simple one in contrast to the rather difficult proof of the converse theorem which Hall had published in the same journal for April, 1927. It seems worthwhile to give a simpler proof of this converse.

**Theorem.** If \( g \) is the order of a soluble group \( G \) and \( m \) is a divisor of \( g \) such that \( m \) and \( g/m \) are relatively prime, then \( G \) has a subgroup of order \( m \) and furthermore all the subgroups of order \( m \) in \( G \) are conjugate under \( G \).

**Proof.** Since the theorem is true by default for prime power groups, let us suppose that it is true for all soluble groups of orders less than \( g \) and use the method of complete induction. Then the theorem is true for an invariant subgroup \( G' \) of prime index \( r \) in \( G \).

If \( r \) divides \( g/m \), then \( G' \) has a subgroup of order \( m \). Since \( G' \) is invariant and \( (m, r) = 1 \), every element of order dividing \( m \) in \( G \) must be in \( G' \). Hence all the subgroups of order \( m \) in \( G \) must be in \( G' \) where by hypothesis they form a complete set of conjugates under \( G' \) and consequently a complete set of conjugates under \( G \).

If \( r \) divides \( m \), then \( G' \) has a complete set of conjugates of order \( m/r \). Since \( G' \) is invariant, the subgroups of order \( m/r \) in \( G' \) are a complete set of conjugates under \( G \). Let \( M' \) be one of these subgroups of order \( m/r \) in \( G' \).

If \( M' \) is the only subgroup of order \( m/r \) in \( G' \), then, it must be invariant in \( G \). Then the quotient group is of order \( rg/m \) and since \( r \) is a Sylow divisor of the order of this quotient group, it has a subgroup of order \( r \). Then \( G \) has a subgroup of order \( r \cdot m/r \) or \( m \). On the other hand if \( M' \) is not the only subgroup of order \( m/r \) in \( G' \), let it be one of \( k \) subgroups of order \( m/r \) in \( G' \). Then \( k \) divides the order \( g/r \) of \( G' \) and the normalizer of \( M' \) in \( G \) is of order \( g/k \) divisible by \( m \). Since this normalizer is a soluble group of order \( g/k \) less than \( g \), it has a subgroup of order \( m \).

There remains only to show, for the case \( r \) divides \( m \), that all the subgroups of order \( m \) in \( G \) are conjugate under \( G \). Let \( M \) be a subgroup of order \( m \). If there is no other subgroup of order \( m \) in \( G \), then the theorem is true by default. However, if \( M_1 \) is another subgroup
of order $m$ in $G$ it will be shown that $M$ and $M_1$ are conjugate under $G$.

Let the crosscut of $M$ and $G'$ be $\Gamma$ of order $\gamma$ and the crosscut of $M_1$ and $G'$ be $\Gamma_1$ of order $\gamma_1$. Then since $G$ is generated by $M$ and $G'$ as well as by $M_1$ and $G'$ and since $G'$ is invariant under both $M$ and $M_1$ we have

$$(g/r)(m)/\gamma = g, \quad (g/r)(m)/\gamma_1 = g,$$

whence

$$\gamma = \gamma_1 = m/r.$$

Then since $\Gamma$ and $\Gamma_1$ are of order $m/r$, they are conjugate under $G'$. If $S^{-1}\Gamma_1S = \Gamma$, then $S^{-1}M_1S$ and $M$ have a common invariant subgroup $\Gamma$. If $\Gamma$ is invariant in $G$, the quotient group $G/\Gamma$ is of order $gr/m$. Since $r$ is a Sylow divisor, the subgroups $M/\Gamma$ and $S^{-1}M_1S/\Gamma$ of order $r$ are conjugate under $G/\Gamma$ and hence $M$ and $S^{-1}M_1S$ are conjugate under $G$ as was to be proved. If, however, $\Gamma$ is not invariant under $G$, its normalizer is a proper subgroup of $G$ containing $M$ and $S^{-1}M_1S$ which are therefore conjugate under the normalizer of $\Gamma$ as was to be proved.

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