POWER SERIES THE ROOTS OF WHOSE PARTIAL SUMS LIE IN A SECTOR

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If the roots of the partial sums of a power series \( f(z) = \sum a_n z^n \) lie in a sector with vertex at the origin and aperture \( \alpha < 2\pi \), the power series cannot have a positive finite radius of convergence. But if \( f(z) \) is an entire function, the roots of its partial sums may lie in such a sector. The question arises: what restrictions are imposed on \( f(z) \) by the requirement that \( \alpha \) be sufficiently small, say \( \alpha < \pi \)? According to a theorem of Pólya the order of \( f(z) \) must be not greater than 1 if the radius of convergence of the power series is positive. Without this assumption the investigation which follows shows that if \( \alpha < \pi \), \( f(z) \) is an entire function of order 0. This result was obtained by Pólya for the case in which \( \alpha = 0 \). 

LEMMA. If the complex numbers \( z_1, \ldots, z_n \) \((z_1 \neq 0)\) lie in a sector with vertex at the origin and aperture \( \alpha < \pi \), then
\[
\frac{n \cos \alpha/2}{\left| \sum_{k=1}^{n} z_k^{-1} \right|} \leq \left| z_1 \cdots z_n \right|^{1/n} \leq \frac{1}{n} \sec \alpha/2 \left| \sum_{k=1}^{n} z_k \right|
\]

When \( \alpha = 0 \) equality occurs if and only if \( z_1 = \cdots = z_n \). When \( \alpha > 0 \) equality occurs if and only if \( n \) is even and \( n/2 \) of the numbers are equal to \( r e^{i\phi} (r > 0; 0 \leq \phi < 2\pi) \) and the other \( n/2 \) numbers are equal to \( r e^{i(\phi + \alpha)} \).

Suppose first that the sector is \(-\alpha/2 \leq \arg z \leq \alpha/2\). Let the \( n \) numbers be
\[
z_k = |z_k| e^{i\theta_k}, \quad -\alpha/2 \leq \theta_k \leq \alpha/2; \ k = 1, \cdots, n.
\]
Since
\[
\sum_{k=1}^{n} z_k = \sum_{k=1}^{n} |z_k| \cos \theta_k + i \sum_{k=1}^{n} |z_k| \sin \theta_k
\]

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Now

\[
\frac{1}{n} \sum_{k=1}^{n} |z_k| \geq |z_1 \cdots z_n|^{1/n}.
\]

Consequently

\[
\frac{1}{n} \sum_{k=1}^{n} z_k \geq \cos \alpha/2 \sum_{k=1}^{n} |z_k|^{1/n}.
\]

Since the numbers $z_1^{-1}, \cdots, z_n^{-1}$ also lie in the sector $-\alpha/2 \leq \text{am } z \leq \alpha/2$, we have

\[
\frac{1}{n} \sum_{k=1}^{n} z_k^{-1} \geq \cos \alpha/2 |z_1^{-1} \cdots z_n^{-1}|^{1/n}.
\]

Combining the last two inequalities, (1) results.

When $\alpha = 0$, (1) reduces to the well known relation among the harmonic, geometric and arithmetic means of $n$ positive numbers. Here equality occurs if and only if $z_1 = \cdots = z_n$.

If equality occurs in (1) it also occurs in (3) and (4). By (4), $|z_1| = \cdots = |z_n|$. By (3), $\cos \theta_k = \cos \alpha/2$; hence $\theta_k = \pm \alpha/2$ ($k = 1, \cdots, n$).

By (2), $\sum_{k=1}^{n} \sin \theta_k = 0$. Therefore if $\alpha > 0$, $n$ must be even, and $n/2$ of the numbers equal $re^{-i\alpha/2}$, while the other $n/2$ numbers equal $re^{i\alpha/2}$. Conversely, when these conditions are satisfied, equality is attained in (1).

If the numbers are in the sector $\phi \leq \text{am } z \leq \text{am } (\phi + \alpha)$, we apply the transformation

\[ z' = e^{-i(\alpha/2 + \phi)}z, \]

which rotates this sector into the sector $-\alpha/2 \leq \text{am } z \leq \alpha/2$ without affecting the value of any member of (1).

**Theorem.** If, for each $n \geq n_0$, the roots of the partial sum of degree $n$ of the formal power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) lie in some sector with vertex at the origin and aperture $\alpha < \pi$, then $f(z)$ is an entire function of order $0$.

The case in which $f(z)$ is a polynomial is trivial and is excluded from

[4] While $\alpha$ is independent of $n$, we do not require that there shall be one sector which contains the roots of all the partial sums of degree $n \geq n_0$; the lines bounding the sector may be different for different values of $n$. 
consideration. We shall suppose $a_0 \neq 0$; otherwise a power of $z$ could be removed from $f(z)$ without affecting the theorem. Let

$$f_n(z) = \sum_{k=0}^{n} a_k z^k,$$

By the Gauss-Lucas theorem the roots of $f_n'(z)$ are also in the sector which contains the roots of $f_n(z)$, and the only roots of $f_n'(z)$ that lie on the boundary of the sector are multiple roots of $f_n(z)$; hence $f_n'(0) \neq 0$. Repeated applications of this argument yield the result that $a_k \neq 0 \ (k = 0, 1, \cdots)$.

According to the lemma, if $z_1, \cdots, z_n$ denote the zeros of $f_n(z)$,

$$|a_n|^{-1/n} \to \infty \quad \text{with} \quad n \to \infty.$$

From the first two members of this inequality it follows that $f(z)$ is an entire function. If $\rho$ is its order,

$$\frac{1}{\rho} = \lim \inf_{n \to \infty} \frac{\log |a_n|^{-1}}{n \log n}.$$

From the last two members of (5) we have

$$\frac{1}{n} \log \frac{1}{|a_n|} - \frac{1}{n-1} \log \frac{|a_{n-1}|}{n(n-1)} \leq \frac{1}{n} \log |a_0| + \frac{1}{n-1} \log nc.$$

Let $m = \max (n_0, 4), n > m$. Substituting $n = m, m+1, \cdots, n$ in (7), and adding, we obtain

$$\frac{1}{n} \log \frac{1}{|a_n|} \geq \frac{1}{m-1} \log \frac{1}{|a_{m-1}|} + \frac{1}{m} \log |a_0| + \frac{1}{m} \log nc.$$

Now

$$\sum_{s=m}^{n} \frac{\log s}{s-1} > \sum_{s=m}^{n} \frac{\log (s-1)}{s-1} > \frac{1}{2} \log^2 n - \frac{1}{2} \log^2 (m-1),$$

$$\sum_{s=m}^{n} \frac{1}{s-1} > \log n - \log (m-1).$$
Consequently

\[
\frac{1}{n} \log \left| \frac{1}{a_n} \right| > A + \frac{1}{2} \log n + \log c \log n,
\]

where \( A \) is bounded as \( n \to \infty \). Comparing (6) and (8), we conclude that \( \rho = 0 \).

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