SPACE CREMONA TRANSFORMATIONS OF ORDER $m+n-1$

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1. Introduction. This paper discusses a space Cremona transformation of order $m+n-1$ ($m, n$ any integers) generated by two rational twisted curves. One special position of the defining curves gives rise to an involution recently described, while another special position results in an involution somewhat similar to one which was defined in a different manner by Montesano.

2. Cremona transformation. Consider a curve $C_n$ of order $n$ having $n-1$ points on each of two skew lines $d$ and $d'$, and a curve $C'_m$ of order $m$ having $m-1$ points on each of $d$ and $d'$ ($m, n$, any integers). A generic point $P$ determines a ray through it intersecting $C_n$ once in $a$ and $d$ once in $\beta$. $P$ also determines a ray through it intersecting $C'_m$ once in $\gamma$ and $d$ once in $\delta$. We define $P'$, the correspondent of $P$, to be the intersection of lines $\alpha\delta$ and $\beta\gamma$.

It is to be noted that if $C_n$ should become identical with $C'_m$ but $d$ and $d'$ remain distinct, there would result the Cremona involution we discussed in a recent paper (loc. cit.).

Let the equations of $d$ be $x_1=0, x_2=0$, and those of $d'$ be $x_3=0, x_4=0$. Let $C_n$ be

\[
x_1 = (as + bt) \prod_{i=1}^{n-1} (t_i - s_i), \quad x_2 = (cs + dt) \prod_{i=1}^{n-1} (t_i - s_i),
\]

\[
x_3 = (es + ft) \prod_{i=n}^{2n-2} (t_i - s_i), \quad x_4 = (gs + ht) \prod_{i=n}^{2n-2} (t_i - s_i),
\]

where $s_i, t_i$ for $i=1, 2, \cdots, n-1$ are values of the parameters of $C_n$ for points on $d$, and for $i=n, n+1, \cdots, 2n-2$, for points on $d'$.

Let the equations of $C'_m$ be

\[
x_1 = (AS + BT) \prod_{i=1}^{m-1} (T_i S - S_i T), \quad x_2 = (CS + DT) \prod_{i=1}^{m-1} (T_i S - S_i T),
\]

\[
x_3 = (ES + FT) \prod_{i=m}^{2m-2} (T_i S - S_i T), \quad x_4 = (GS + HT) \prod_{i=m}^{2m-2} (T_i S - S_i T),
\]

1 Presented to the Society, September 10, 1940.


where \( S_i, T_i \) for \( i = 1, 2, \ldots, m - 1 \) are values of the parameters of \( C'_m \) for points on \( d \), and for \( i = m, m+1, \ldots, 2m-2 \), for points on \( d' \). Then the equations of the transformation are

\[
x_1' = k(Q_1 x_3 + Q_2 x_4) \left( \prod_{i=1}^{n-1} \theta_i \right) \left( \prod_{i=1}^{m-1} \Phi_i \right),
\]

\[
x_2' = k(R_1 x_3 + R_2 x_4) \left( \prod_{i=1}^{n-1} \theta_i \right) \left( \prod_{i=1}^{m-1} \Phi_i \right),
\]

\[
x_3' = k'(r_2 x_1 - q_2 x_2) \left( \prod_{i=n}^{2n-2} \theta_i \right) \left( \prod_{i=m}^{2m-2} \Phi_i \right),
\]

\[
x_4' = k'(q_1 x_2 - r_1 x_1) \left( \prod_{i=n}^{2n-2} \theta_i \right) \left( \prod_{i=m}^{2m-2} \Phi_i \right),
\]

where \( k = (bc-ad), \) \( K' = (FG-EH) \), and

\[
Q_1 = (AH-BG), \quad Q_2 = (BE-AF),
\]

\[
R_1 = (CH-DG), \quad R_2 = (DE-CF),
\]

\[
q_1 = (ah-bg), \quad q_2 = (be-af),
\]

\[
r_1 = (ch-dg), \quad r_2 = (de-cf),
\]

\[
\theta_i = \{ t_i(bx_2 - dx_1) - s_i(cx_1 - ax_2) \},
\]

\[
\Phi_i = \{ T_i(H x_3 - F x_4) - S_i(E x_4 - G x_3) \}.
\]

The inverse transformation is

\[
x_1 = K(q_1 x_3' + q_2 x_4') \left( \prod_{i=1}^{n-1} \phi_i \right) \left( \prod_{i=1}^{m-1} \Theta_i \right),
\]

\[
x_2 = K(r_1 x_3' + r_2 x_4') \left( \prod_{i=1}^{n-1} \phi_i \right) \left( \prod_{i=1}^{m-1} \Theta_i \right),
\]

\[
x_3 = k'(R_2 x_1' - Q_2 x_2') \left( \prod_{i=n}^{2n-2} \phi_i \right) \left( \prod_{i=m}^{2m-2} \Theta_i \right),
\]

\[
x_4 = k'(Q_1 x_2' - R_1 x_1') \left( \prod_{i=n}^{2n-2} \phi_i \right) \left( \prod_{i=m}^{2m-2} \Theta_i \right),
\]

where \( K = (BC-AD), \) \( k' = (fg-eh) \),

\[
\phi_i = \{ t_i(h x_4' - f x_3') - s_i(e x_3' - g x_4') \},
\]

\[
\Theta_i = \{ T_i(D x_1' - B x_2') - S_i(A x_3' - C x_4') \}.
\]

Both the direct and inverse transformations are of order \( m+n-1 \), where \( m \) and \( n \) are any integers.

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The fundamental system and its images for the direct transformation are as follows.

d is an \((n-1)\)-fold \(F\)-line of simple contact. The fixed tangent planes are \(\theta_i = 0\), where \(i = 1, 2, \ldots, n-1\). It is of the first species and its \(P\)-surface consists in the planes \(\phi_i' = 0\), where \(i = 1, 2, \ldots, n-1\), which pass through \(d'\).

d' is an \((m-1)\)-fold \(F\)-line of simple contact. The fixed tangent planes are \(\Phi_i = 0\), where \(i = m, m+1, \ldots, 2m-2\). It is of the first species and its \(P\)-surface consists in the \(m-1\) planes \(\Theta_i' = 0\) through \(d\), where \(i = m, m+1, \ldots, 2m-2\).

Each of the \(m-1\) intersections of \(C_m\) and \(d\) is an \(n\)-fold isolated \(F\)-point. Their \(P\)-surfaces are \(\Theta_i' = 0\), where \(i = 1, 2, \ldots, m-1\), respectively.

Each of the \(n-1\) intersections of \(C_n\) and \(d'\) is an \(m\)-fold isolated \(F\)-point. Their \(P\)-surfaces are \(\phi_i' = 0\) (\(i = n, n+1, \ldots, 2n-2\)) respectively.

The \((n-1)(m-1)\) lines of intersection of the \(n-1\) fixed tangent planes through \(d\) with the \(m-1\) fixed tangent planes through \(d'\) are simple \(F\)-lines without contact. They are of the second species.

The \((m-1)(n-1)\) lines joining the \(m-1\) \(n\)-fold isolated \(F\)-points on \(d\) with the \(n-1\) \(m\)-fold isolated \(F\)-points on \(d'\) are simple \(F\)-lines without contact. They are of the second species.

We may obtain a description of the fundamental system of the inverse transformation by interchanging \(m\) and \(n\), \(C_n\) and \(C_m\), \(\theta_i\) and \(\Theta_i'\), \(\Phi_i\) and \(\phi_i'\), wherever they appear in the foregoing.

\(C_n\), \(d\), and \(d'\) lie on the same quadric surface \(Q\), and \(C_m'\), \(d\), and \(d'\) lie on a quadric surface \(Q'\). These quadrics may be the same or distinct and, while this does not affect the preceding discussion, the invariant systems for the two cases are different.

When \(Q\) and \(Q'\) are distinct, they intersect in \(d\), \(d'\), and two transversals \(l_1\) and \(l_2\). The \(d\) and \(d'\) are common generators of the \(\mu\)-systems of the two quadrics, while \(l_1\) and \(l_2\) are common generators of their \(\lambda\)-systems. The transformation sends each \(\lambda\)-generator of \(Q\) over into a \(\lambda\)-generator of \(Q'\), and each \(\lambda\)-generator of \(Q'\) over into a \(\lambda\)-generator of \(Q\). Thus \(Q\) as a whole corresponds to \(Q'\) and vice versa. Each \(\lambda\)-generator of either quadric belongs to a cycle of index four—that is, four applications of the transformation leave every \(\lambda\)-generator invariant. The transformation interchanges \(C_n\) and \(C_m'\). The points of \(l_1\) are in involution; thus \(l_1\) is an invariant line and the two fixed points of the involution are invariant points. Similarly for \(l_2\). These four invariant points are the only invariant points that are not also \(F\)-points.

Let us now consider the case where \(C_n\), \(C_m'\), \(d\), and \(d'\) all lie on the
same quadric \( Q = x_1x_4 - x_2x_3 = 0 \). The transformation causes \( C_n \) and \( C_m' \) to interchange. The pencil of planes \( x_3 - \lambda x_1 = 0 \) is in involution with the pencil \( x_3 - \lambda x_1 = 0 \) and this makes each \( \lambda \)-generator of \( Q \) invariant. Consequently \( Q \) is invariant. The locus of invariant points is a curve \( K_{m+n} \) of order \( m+n \) lying on \( Q \). \( K_{m+n} \) passes through the \( m+n-2 \) points of intersection of \( C_n \) and \( C_m' \) and intersects \( d \) and \( d' \) in the \( m+n-2 \) isolated \( F \)-points on each of them. It intersects every \( \mu \)-generator of \( Q \) in \( m+n-2 \) points and intersects every \( \lambda \)-generator in two points.

3. **Involution.** Consider a twisted curve \( C_n \) having \( n-1 \) points \( \Delta_i \) on a straight line \( d \), and a curve \( C_m' \) having \( m-1 \) points \( \Sigma_i \) on the same straight line \( d \) (\( m, n \) any integers). A generic point \( P \) determines a ray through it intersecting \( C_n \) in \( \alpha \) and \( d \) in \( \beta \), and also a ray through it intersecting \( C_m' \) in \( \gamma \) and \( d \) in \( \delta \). We define \( P' \), the correspondent of \( P \) in the involution, to be the intersection of lines \( \alpha \delta \) and \( \beta \gamma \).

If, in §2, we make \( d \) and \( d' \) identical, we obtain an involution of this kind. However, the curves \( C_n \) and \( C_m' \) of the present section do not necessarily lie on quadric surfaces.

The involution is of order \( m+n-1 \).

The fundamental system and its principal images follow.

\( d \) is an \( (n+m-2) \)-fold \( F \)-line of simple contact. The fixed tangent planes are \( \theta_i = 0 \), where \( i = 1, 2, \ldots, n-1 \), and \( \Theta_i = 0 \), where \( i = 1, 2, \ldots, m-1 \). It is of the second species and counts \( (n+m-1) \cdot (n+m-2) \) times in the intersection of any two homaloids.

Points \( \Delta_i \) are isolated \( F \)-points. Their \( P \)-surfaces are the planes \( \theta_i = 0 \) \( (i = 1, 2, \ldots, n-1) \) respectively.

Points \( \Sigma_i \) are isolated \( F \)-points. The \( P \)-element of each is \( \Theta_i = 0 \) \( (i = 1, 2, \ldots, m-1) \) respectively.

As we have seen, a general point \( P \) determines with \( d \) a plane \( \pi \) intersecting \( C_n \) in \( \alpha \) and \( C_m' \) in \( \gamma \). Call \( L \) the intersection of lines \( \alpha \gamma \) and \( d \). Then \( J \), the harmonic conjugate of \( L \) with respect to \( \alpha \) and \( \gamma \), will be the only invariant point of \( \pi \) other than points of \( d \). As \( \pi \) makes one revolution about \( d \), \( \alpha \) moves in \( \pi \) and crosses \( d \) \( n-1 \) times; \( \gamma \) also moves in \( \pi \), crossing \( d \) \( m-1 \) times. As \( \alpha \) approaches \( d \), \( J \) approaches the same point on \( d \), and the locus of \( J \) intersects \( d \) in all the points \( d \) has in common with \( C_n \) and \( C_m' \). The locus of \( J \) is a rational curve \( K_{m+n-1} \) of order \( m+n-1 \) having \( m+n-2 \) points on \( d \). \( K_{m+n-1} \) is the locus of invariant points.

It is clear that the line \( PP' \) intersects \( K_{m+n-1} \) in \( J \) and \( d \) in \( L \), and that \( P \) and \( P' \) are harmonic conjugates with respect to \( d \) in \( L \).

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4 Compare with Montesano, loc. cit.
4. **Lower order for particular positions of the defining elements.** Each of the fixed tangent planes $\theta_i = 0$ mentioned in the contact conditions for the involution passes through $d$ and is tangent to $C_n$ at the corresponding $\Delta_i$. The fixed tangent planes $\Theta_i = 0$ are similarly related to the curve $C_m'$.

If $C_n$ and $C_m'$ are so situated that a plane of $\theta_i = 0$ ($i = 1, 2, \cdots, n - 1$) coincides with a plane of $\Theta_i = 0$ ($i = 1, 2, \cdots, m - 1$), the order of the involution is reduced by one. In this way we may reduce the order by any integer up to, and including, the smaller of the two numbers $n - 1$ and $m - 1$.

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