NOTE ON AUTOPOLAR CURVES[1]

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1. **Introduction.** The aim of this paper is to study those curves which are autopolar with respect to the parabola $2\eta=\xi^2$. The method, which is believed to be new, is to consider these curves as special solutions of those differential equations which are invariant under the dual substitutions for the above conic of reference.[2] It will be obvious that this method may be readily modified for the study of curves which are autopolar with respect to any conic. The parabola $2\eta=\xi^2$ has been chosen for the sake of the simplicity of the substitutions.

2. **Dual substitutions for the conic of reference.** In the ordinary differential equation,

$$f(x, y, y', y'', \cdots, y^{(n)}) = 0, \tag{1}$$

let us make the well known dual substitutions[3]

$$\begin{aligned} x = P = Y', \qquad y = Y'X - Y, \qquad p = y' = X,\\ y'' = 1/Y'', \ y''' = -Y'''/Y''^3, \cdots \end{aligned} \tag{2}$$

for which the conic of reference is the parabola $2\eta=\xi^2$. We obtain a new differential equation,

$$f(Y', Y'X - Y, X, 1/Y'', \cdots) = 0, \tag{3}$$

whose solution is, let us say,

$$\phi(X, Y, c_1, c_2, \cdots, c_n) = 0. \tag{4}$$

If we eliminate X, Y from equation (4) and the following two equations,[4]

$$x\frac{\partial\phi}{\partial Y}+\frac{\partial\phi}{\partial X}=0, \qquad -y\frac{\partial\phi}{\partial Y}=Y\frac{\partial\phi}{\partial Y}+X\frac{\partial\phi}{\partial X}, \tag{5}$$

we shall have the solution of the original differential equation (1), which we shall denote by

$$F(x, y, c_1, c_2, \cdots, c_n) = 0. \tag{6}$$

3. **Geometrical interpretation.** Let C be any curve of the family

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[1] Presented to the Society, October 28, 1939.
[2] A. R. Forsyth, *A Treatise on Differential Equations*, 3d edition, 1903, pp. 45–47.
[3] Forsyth, op. cit.
[4] Forsyth, loc. cit.

(6). As a point P traverses C, the polar of P relative to the conic of reference will envelop some member C' of the family (4), and vice versa. That is, C and C' are polar reciprocals.

If (1) is of the first order, equations (1) and (3) may have singular solutions. If E and E' denote the envelopes of the families (4) and (6), it is evident from the above that E and E' are also polar reciprocals.

In addition to these relations between the families (4) and (6), there are several well known relations between the extraneous loci which may exist in connection with the integral curves. For example, a cuspidal locus for the integral curves (4), [(6)], will correspond to the locus of points of inflexion for the integral curves (6), [(4)].[5]

4. **Condition that $y=f(x)$ be autopolar.** If a curve C, $y=f(x)$, be autopolar, that is, its own polar reciprocal, the polar of any point (x_1, y_1) on C will be tangent to the curve at some other point (x_2, y_2). Relative to the above conic of reference, the polar of any point (x, y) on C is $x\xi-\eta-y=0$, which we may write

$$x\xi - \eta - f(x) = 0. \tag{7}$$

This is a one-parameter family of lines with x as the parameter, and their envelope will be found by the elimination of x from (7) and the following equation,

$$\xi - f'(x) = 0. \tag{8}$$

The necessary and sufficient condition that C be autopolar is that on eliminating x we shall get $\eta=f(\xi)$ as the envelope. From (7) and (8) we have $xf'(x)-\eta-f(x)=0$; and on replacing η by $f(\xi)$, or $f[f'(x)]$, we have

$$f[f'(x)] + f(x) - xf'(x) = 0. \tag{9}$$

We have, therefore, the following theorem:

THEOREM 1. *A necessary and sufficient condition that a curve $y=f(x)$ be autopolar with respect to the conic $2\eta=\xi^2$ is that the relation* (9) *be satisfied.*

It is interesting to note that (9) is of the Clairaut type.

5. **Relations between conjugate pairs,** (x_1, y_1) **and** (x_2, y_2). Consider any point $P_1(x_1, y_1)$ on an autopolar curve C. The polar of P_1 is $x_1\xi-\eta-y_1=0$; and since this line is tangent to C at some point

[5] Sophus Lie and Georg Scheffers, *Geometrie der Berührungstransformationen*, vol. 1, 1896, pp. 24–27.

$P_2(x_2, y_2)$, the equation of the polar of P_1 must be satisfied by the coördinates of P_2. Consequently,[6]

$$y_1 + y_2 = x_1 x_2. \tag{10}$$

From the theory of polar reciprocals we know that the polar of P_2 is tangent to C at P_1, and since the slopes of the polars of P_1 and P_2 are respectively x_1 and x_2, we have

$$x_1 = f'(x_2), \qquad x_2 = f'(x_1). \tag{11}$$

Let us call such a pair of points, P_1 and P_2, a conjugate pair, and say that each is the conjugate of the other. The relations (11) are also evident from the set of substitutions (2) in which $x = P$, $X = p$.

From (10) and (11) we have the following theorem:

THEOREM 2. *On any autopolar curve with respect to the conic* $2\eta = \xi^2$, *the product of the slopes at a conjugate pair is equal to the sum of the ordinates at these points.*

6. **Self-conjugate points.** For any real conjugate pair, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, let us suppose $x_2 > x_1$. Let us also assume that $f(x)$ is defined for all values of x between x_1 and x_2. Now let x_2 decrease, that is, let P_2 approach the original position of P_1. When P_2 arrives at this position. P_1 will have arrived at the original position of P_2. Hence x_1 must increase as x_2 decreases. Consequently, between each real conjugate pair there exists a point $P_3(x_3, y_3)$ which is self-conjugate. From (10) we shall have $2y_3 = x_3^2$, and therefore P_3 lies on the conic of reference. Since the polar of P_3 is tangent to the conic of reference at this point, we see that every autopolar curve has a common tangent with this conic.

Also from (11), and the law of the mean, we have

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} = -1 = f''(x_4), \qquad x_1 < x_4 < x_2, \tag{12}$$

and since this relation is satisfied by every conjugate pair, no matter how near they may be to P_3, we see that

$$f''(x_3) = -1. \tag{13}$$

From (11) and (13) we readily find the curvature of any autopolar curve at P_3 to be

$$K = \frac{-1}{[1 + f'^2(x_3)]^{3/2}} = \frac{-1}{(1 + x_3^2)^{3/2}}.$$

[6] It is obvious from (2) that (10) is simply a restatement of (9).

It is readily verified that, except for sign, this is the curvature of the conic of reference at P_3. Hence we have:

THEOREM 3. *For every autopolar curve with respect to the conic* $2\eta=\xi^2$, *the curvature at the self-conjugate points* P_3 *is the same, except for sign, as the curvature of the conic at* P_3.

We note that this theorem applies also to the case of the general conic by virtue of the fact that under any collineation the ratio of the curvatures of two curves at a point at which they are tangent is invariant.[7]

From (12) it also follows that between real conjugate pairs $f'(x)$ is a decreasing function.

7. **Loci associated with a conjugate pair.** Consider the locus of Q, the intersection of the polars for any conjugate pair, P_1 and P_2. The coördinates (ξ, η) of Q are readily found from the equations of the polars of P_1 and P_2; we find

$$\xi = \frac{y_1 - y_2}{x_1 - x_2}, \qquad \eta = \frac{x_2y_1 - x_1y_2}{x_1 - x_2}. \tag{14}$$

By means of (11) these equations of the locus of Q are readily given in terms of one parameter, say x_1.

Since the polar of Q is the line P_1P_2, we see that the locus of Q and the envelope of the lines P_1P_2 are also polar reciprocals.

It will be of interest to consider also the locus of S, the mid-point of the segment P_1P_2.

We shall also consider the locus of R, the intersection of normals at conjugate pairs. The coördinates of R are readily found to be

$$\xi = \frac{x_1^2 + x_1x_2y_1 - x_2^2 - x_1x_2y_2}{x_1 - x_2}, \quad \eta = \frac{x_2 + x_1y_2 - x_1 - x_2y_1}{x_1 - x_2}. \tag{15}$$

From §6 it is evident that the loci of Q and S must pass through P_3.

8. **Differential equations invariant under the above transformation.** Any differential equation of the form

$$f(x)y'' + f(y') = 0 \tag{16}$$

becomes, on using (2), $f(X)Y''+f(Y')=0$, which is identical with (16). Let us denote the solution of (16) by

[7] G. Fubini and E. Čech, *Introduction à la Géométrie Protective Différentielle des Surfaces*, 1931, pp. 17–20.

$$\phi(x, y, c_1, c_2) = 0. \tag{17}$$

As a point P traverses some member C of (17), the polar of P will, in general, envelop some other member C' of the same family. We wish to determine if there are any members of the two-parameter family (17), which are autopolar. The method for the determination of these curves will be illustrated by particular examples.

EXAMPLE 1. $(x+1)y''+y'+1=0$. The solution of this is

$$y = c_1 \log (x + 1) - x + c_2. \tag{18}$$

Here $f(x)=c_1 \log (x+1)-x+c_2$, and $f'(x)=c_1/(x+1)-1$. If we put these expressions in (9), we get $(c_1-c_1 \log c_1-1-2c_2)x+c_1-c_1 \log c_1 -1-2c_2=0$, which is satisfied only if $c_2=\frac{1}{2}c_1-\frac{1}{2}c_1 \log c_1-\frac{1}{2}$. Consequently, of the two-parameter family of curves (18), we have the following one-parameter family whose members are autopolar:

$$y = c_1 \log (x + 1) - x + \tfrac{1}{2}c_1 - \tfrac{1}{2}c_1 \log c_1 - \tfrac{1}{2}. \tag{19}$$

The particular member of (19) for which $c_1=1$ has the property that the locus of S (the mid-point of P_1P_2) is the line $x+y=0$.

It may be readily verified that, as indicated in §6, the conic of reference, $2\eta=\xi^2$, is the envelope of the one-parameter family of autopolar curves (19).

EXAMPLE 2. $x^3y''+y'^3=0$. The solution is $c_1^2(y-c_2)^2=c_1x^2-1$; and on putting this in (9) we have $2c_2(c_1x^2-1)^{1/2}=0$. Hence $c_2=0$, and of this two-parameter family of hyperbolas there exists the one-parameter autopolar family $c_1x^2-c_1^2y^2=1$. It is readily found from (14) that the locus of Q for these curves is the line $\eta=1/c_1$. And from (15) the locus of R is the line $\eta=-(c_1+1)/c_1$.

EXAMPLE 3. $y''+1=0$. The solution is $y=-\frac{1}{2}x^2+c_1x+c_2$, and this in (9) gives $c_2=-\frac{1}{4}c_1^2$. Hence the parabolas $y=-\frac{1}{2}x^2+c_1x-\frac{1}{4}c_1^2$ are autopolar. It is readily found that for these curves the locus of Q is the line $\xi=\frac{1}{2}c_1$, and that this line is also the locus of S. We also find that the coördinates of the mid-point of the segment QS, $(\frac{1}{2}c_1, \frac{1}{8}c_1^2)$, are identical with the coördinates of P_3, the point of contact of the parabolas with the conic of reference.

EXAMPLE 4. $x+y'=xy'$. The solution of this first-order equation is $y=x+\log (x-1)+c$, and this in (9) gives $c=0$. Hence there is but one member of the family which is autopolar.

EXAMPLE 5. $y'y''+x=0$. From this equation we derive the following one-parameter family of autopolar curves:

$$y = \tfrac{1}{2}c_1^2 \arcsin x/c_1 + \tfrac{1}{2}x(c_1^2 - x^2)^{1/2} - \tfrac{1}{8}c_1^2\pi.$$

There are, of course, many other types of differential equations which are invariant under (2). We mention a few of these: $f(x)+f(y')=F(xy')$, $2y-xy'=f(y')-f(x)$, and $f(y)y''+f(xy'-y)=0$.

The above examples suggest that, in general, we have the following distinction between the autopolar curves obtained from invariant differential equations of the first and second orders. When the solution of such a first-order equation is put in (9), this relation will be satisfied by only a finite number of values of the one arbitrary constant; and hence of this single infinity of curves we shall have but a finite number which are autopolar. When the solution of an invariant second-order equation is put in (9), the relation will be satisfied in many cases by some relation between the two arbitrary constants; that is, from the double infinity of solutions we select a single infinity of autopolar curves. Moreover, in every case, this one-parameter family of autopolar curves envelopes the conic of reference $2\eta=\xi^2$.

There are, however, exceptions to the above. For example, consider the differential equation $(y-x^2)y'+xy=0$, invariant under (2), whose solution is $cx^2+y^2-2cy=0$. On using (9) we find there is no value of c which satisfies this relation. That is, among these conics there are no autopolar curves; but for each conic of the family the polar reciprocal is some other conic of the same family. It may readily be shown that of the above conics, those for which the values of c are negatives of one another are polar reciprocals.

9. **Other dual substitutions.** The dual substitutions for any given conic of reference are easily derived after the manner outlined in Forsyth. And any differential equation which is invariant under these substitutions will have among its solutions certain curves which are autopolar with respect to the given conic.

For example, if the conic of reference be the unit circle, these dual substitutions are[8]

$$y' = -\frac{X}{Y}, \quad x = \frac{-Y'}{Y-XY'}, \quad y = \frac{1}{Y-XY'}, \cdots. \tag{20}$$

Without repeating the argument of §4, the necessary and sufficient condition that a curve $y=F(x)$ be autopolar with respect to the unit circle is

$$\frac{1}{f(x)-xf'(x)} = F\left[\frac{f'(x)}{xf'(x)-f(x)}\right]. \tag{21}$$

[8] Lie und Scheffers, loc. cit., p. 23.

EXAMPLE 6. $xy'+y=0$. This equation is invariant under (20); and its solution, $y=c/x$, when put in (21), satisfies this relation when $c=\pm\frac{1}{2}$. Consequently, the equilateral hyperbolas, $2xy=\pm 1$, are autopolar with respect to the unit circle.

For a complete bibliography on the subject of autopolar curves the reader is asked to consult H. Brocard and T. Lemoyne, *Courbes Géométriques Remarquables*, vol. 1, 1919, pp. 430–432.

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