MOMENT PROBLEM FOR A BOUNDED REGION

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1. Introduction. In this paper a solution of the moment problem given by Hausdorff\(^2\) for a bounded interval is extended to any bounded region in euclidean \(n\)-space, under certain conditions on polynomial expansions over the region. The resulting solution is valid for the \(n\)-dimensional sphere, and includes the Hausdorff case as well as the known conditions on the “class” of Fourier and Fourier-Stieltjes series.\(^3\)

2. Definitions and notation. Let \(n\) be a positive integer, fixed but arbitrary. \(R^n\) will denote the euclidean \(n\)-space, \((x)\) and \((y)\) will stand for \((x_1, x_2, \cdots, x_n)\) and \((y_1, y_2, \cdots, y_n)\), points of \(R^n\), and \(E\) a bounded, closed subset of \(R^n\). \(v, \tau, i, j, k, \) and \(s\), will be used for non-negative integers, and \((k), (s), \) and so on, will denote ordered \(n\)-tuples of non-negative integers \((k_1, k_2, \cdots, k_n), (s_1, s_2, \cdots, s_n)\), and so on, \((k) = (s)\) will mean \(k_i = s_i, \) \(i = 1, 2, \cdots, n\). \((0)\) will mean \(0, 0, \cdots, 0\), \{\(\mu_{(m)}\)\} will be a sequence of real numbers, and \{\(U_{(k)}(x)\)\} and \{\(V_{(k)}(x)\)\} will be two sequences of polynomials such that

\[
U_{(0)}(x) = V_{(0)}(x) = \text{const.},
\]

\[
\int_E U_{(k)}(x)V_{(s)}(x)dx = \begin{cases} 0, & (k) \neq (s), \\ 1, & (k) = (s), \end{cases}
\]

and by \(\int_E f(x, y)d\Phi(E)\) will be meant the Lebesgue-Stieltjes integral over \(E\) of \(f\) considered as a function of a point \((y)\). \(B\) will be used for any Borel set with \(B \subseteq E\).

If \(f\) is integrable over \(E\) we define

\[
\mathcal{S}(f, x) \simeq \sum_{(k)} A_{(k)} V_{(k)}(x), \quad A_{(k)} = \int_E f(x) U_{(k)}(x)dx,
\]

\[
S(x, y) \simeq \sum_{(k)} U_{(k)}(x)V_{(k)}(y).
\]

Let \(L_v\) for every \(v\) be a partition of \(R^n\) into two subsets, one closed and bounded. We write \((k) \in L_v\) to indicate that \((k)\) belongs to the

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\(^1\) Presented to the Society, June 20, 1940.


\(^3\) See, for example, A. Zygmund, *Trigonometrical Series*, Monografje Matematyczne, vol. 5, Warsaw, 1935, pp. 79–86.

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bounded subset defined by $L_r$, and require that for every $(k)$ there exist a $v$ such that $(k) \in L_r$, and that $(k) \in L_r$ shall imply $(k) \in L_v$, for all $v' \geq v$. Now let

$$S_v(x, y) = \sum_{(k) \in L_v} U_{(k)}(x)V_{(k)}(y),$$

$$\mathcal{S}_v(f, x) = \sum_{(k) \in L_v} A_{(k)} V_{(k)}(x) = \int_E S_v(x, y)f(y)dy.$$ 

If $T: \mathbb{R}^{dtl} \rightarrow \mathbb{R}^{dtl}$ is any regular Toeplitz transformation, we write

$$T \mathcal{S}_v(f, x) = \int_E T S_v(x, y)f(y)dy = \int_E K_v(x, y)f(y)dy.$$ 

If $P$ is a polynomial in $(x)$ we denote by $\mu_{(m)}(P)$ the expression resulting from the substitution of $x_1^{m_1}x_2^{m_2} \cdots x_n^{m_n}$ in $P$.

3. **Moment problem.** A solution of the moment problem for the set $E$ is given in the following theorem:

**Theorem.** Given $\{U_{(k)}(x)\}$, $\{V_{(k)}(x)\}$, $\{L_v\}$, and $T$ satisfying the conditions above, and such that $TS_v(x, y)=K_v(x, y) \geq 0$ for all $(x)$, $(y) \in E$, and all $v$, and such that for any $f$ integrable over $E$ $T \mathcal{S}_v(f, x) = f(x)$ for every $(x) \in E$ for which $f$ is continuous, and uniformly on $E$ if $f$ is continuous on $E$, then in order that a sequence $\{\mu_{(m)}\}$ be expressible in the form

$$\mu_{m_1, m_2, \ldots, m_n} = \int_E x_1^{m_1}x_2^{m_2} \cdots x_n^{m_n} d\Phi(E),$$

where $\Phi$ is completely additive, defined over at least all Borel sets of $R^n$, and with

1. $\int_E d\Phi(E) \leq M$,
2. $\Phi(B) \geq 0$,
3. $\Phi(B) = \int_B \phi(x)dx$ and with
   - $\phi \in L^p_B$, $p > 1$,
   - $\phi \in L_B$,
   - $|\phi| \leq M$,
   - $\phi \in C_B$,

it is necessary and sufficient that

1. $\int_E |K_v(x, y)| dx \leq M$ for all $v$,
2. $\mu_{(v)} \{K_v(x, y)\} \geq 0$ for all $(x) \in E$ and all $v$,
3. $\int_E |\mu(x, y)|^p dx \leq M$ for all $v$,

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* Zygmund, loc. cit., pp. 40–43.
(3b) \[ \lim_{r \to \infty} \int_{E} |\mu(y) \{ K_r(x, y) \} - \mu(y) \{ K_r(x, y) \}| \, dx = 0, \]
(3c) \[ |\mu(y) \{ K_r(x, y) \}| \leq M \text{ for all } (x) \in E \text{ and all } v, \]
(3d) \[ \lim_{r \to \infty} |\mu(y) \{ K_r(x, y) \} - \mu(y) \{ K_r(x, y) \}| = 0 \text{ uniformly in } (x) \in E. \]

The proof in each of the six cases closely parallels that of Hausdorff. The proof is given for case (1) to indicate the modifications:

**Necessity.** We have

\[ |\mu(y) \{ K_r(x, y) \}| = \left| \int_{E(y)} K_r(x, y) \, d\Phi(E) \right| \]
\[ \leq \int_{E(y)} K_r(x, y) \, d\Phi(E), \]
\[ \int_{E} |\mu(y) \{ K_r(x, y) \}| \, dx \leq \int_{E} \left\{ \int_{E} K_r(x, y) \, dx \right\} \, d\Phi(E) \]
\[ \leq C \int_{E} |d\Phi(E)| \leq M \]

for

\[ K_r(x, y) = \sum_{j=0}^{\infty} a_{rj} \sum_{(k) \in L_j} U_{(k)}(x)V_{(k)}(y) \]
\[ \int_{E} K_r(x, y) \, dx = \sum_{j=0}^{\infty} a_{rj} \sum_{(k) \in L_j} V_{(k)}(y) \int_{E} U_{(k)}(x) \, dx \]
\[ = \sum_{j=0}^{\infty} a_{rj} \leq \sum_{j=0}^{\infty} |a_{rj}| \leq C. \]

**Sufficiency.** Let

\[ \Phi_r(B) = \int_{B} \mu(y) \{ K_r(x, y) \} \, dx, \]
\[ \int_{E} |d\Phi_r(E)| = \int_{E} |\mu(y) \{ K_r(x, y) \}| \, dx \leq M \]

and, by a well known theorem of Helly, there is a subsequence \( \{ \Phi_r \} \) and a function \( \Phi \) such that \( \int_{E} |d\Phi(E)| \leq M \) and \( \Phi_r(B) \to \Phi(B) \), and also \( \int_{E} V_{(k)}(y) \, d\Phi_r(E) \to \int_{E} V_{(k)}(y) \, d\Phi(E) \) whence \( \mu(y) \{ V_{(k)}(y) \} = \int_{E} V_{(k)}(y) \, d\Phi(E) \), and \( \Phi \) is a solution.

4. **Examples and conclusion.** If \( E \) is the unit sphere in \( R^n \), \( \{ U_{(k)}(x) \} \) and \( \{ V_{(k)}(x) \} \) may be taken as the normalized polynomials of Appell-
Didon,\(^5\) \(\sum_{i=1}^{\infty} k_i \leq \nu\), and \(T\) any \((C, r)\) with \(r \geq n + 1\).\(^6\) In particular, for \(n = 1\) this reduces to the Hausdorff solution for the unit interval. If \(E\) is the circumference of the unit circle we may set \(U_0(x) = V_0(x) = (2\pi)^{-1/2}\), and, for \(k > 0\),

\[
U_{2k}(x) = V_{2k}(x) = (\pi)^{-1/2} \cos k\theta, \quad U_{2k-1}(x) = V_{2k-1}(x) = (\pi)^{-1/2} \sin k\theta
\]

with \((s) \in L_\nu\) meaning \(s \leq 2\nu\), \(T\) any \((C, r)\) with \(r \geq 1\).\(^7\) Sequences \(\{U_b(x)\}\) and \(\{V_b(x)\}\) can be constructed by the Schmidt process for any bounded region in \(\mathbb{R}^n\). It would be interesting to know whether regular Toeplitz transformations of the type required for the present theorem exist in general.

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\(^7\) L. Fejer's theorem. See, for instance, Zygmund, loc. cit., p. 45.