MOMENT PROBLEM FOR A BOUNDED REGION¹

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1. Introduction. In this paper a solution of the moment problem given by Hausdorff² for a bounded interval is extended to any bounded region in euclidean n-space, under certain conditions on polynomial expansions over the region. The resulting solution is valid for the n-dimensional sphere, and includes the Hausdorff case as well as the known conditions on the “class” of Fourier and Fourier-Stieltjes series.³

2. Definitions and notation. Let n be a positive integer, fixed but arbitrary. \( R^n \) will denote the euclidean n-space, \((x)\) and \((y)\) will stand for \((x_1, x_2, \ldots, x_n)\) and \((y_1, y_2, \ldots, y_n)\), points of \( R^n \), and \( E \) a bounded, closed subset of \( R^n \). \( v, \tau, i, j, k, \) and \( s \), will be used for non-negative integers, and \((k), (s)\), and so on, will denote ordered \( n \)-tuples of non-negative integers \((k_1, k_2, \ldots, k_n)\), \((s_1, s_2, \ldots, s_n)\), and so on, \((k) = (s)\) will mean \( k_i = s_i, i = 1, 2, \ldots, n \). \((0)\) will mean \((0, 0, \ldots, 0)\), \( \{\mu(m)\} \) will be a sequence of real numbers, and \( \{U(k)(x)\} \) and \( \{V(k)(x)\} \) will be two sequences of polynomials such that

\[ U(0)(x) = V(0)(x) = \text{const.,} \]

(1)

\[ \int_E U(k)(x)V(s)(x)dx = \begin{cases} 0, & (k) \neq (s), \\ 1, & (k) = (s), \end{cases} \]

and by \( \int_E f(x, y)d\Phi(E) \) will be meant the Lebesgue-Stieltjes integral over \( E \) of \( f \) considered as a function of a point \((y)\). \( B \) will be used for any Borel set with \( B \subseteq E \).

If \( f \) is integrable over \( E \) we define

\[ \mathbb{S}(f, x) \simeq \sum_{(k)} A(k)V(k)(x), \quad A(k) = \int_E f(x)U(k)(x)dx, \]

\[ S(x, y) \simeq \sum_{(k)} U(k)(x)V(k)(y). \]

Let \( L_v \) for every \( v \) be a partition of \( R^n \) into two subsets, one closed and bounded. We write \((k) \in L_v \) to indicate that \((k)\) belongs to the

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bounded subset defined by \( L_r \), and require that for every \( (k) \) there exist a \( v \) such that \( (k) \subseteq L_r \), and that \( (k) \subseteq L_v \) shall imply \( (k) \subseteq L_v \) for all \( v' \geq v \). Now let

\[
S_v(x, y) = \sum_{(k) \in L_v} U_{(k)}(x)V_{(k)}(y),
\]

\[
\mathbb{S}_v(f, x) = \sum_{(k) \in L_v} A_{(k)}V_{(k)}(x) = \int_E S_v(x, y)f(y)dy.
\]

If \( T: \|a_{ij}\| \) is any regular Toeplitz transformation,\(^4\) we write

\[
T\mathbb{S}_v(f, x) = \int_E TS_v(x, y)f(y)dy = \int_E K_v(x, y)f(y)dy.
\]

If \( P \) is a polynomial in \( (x) \) we denote by \( \mu_{(m)}(P) \) the expression resulting from the substitution of \( x_1^{m_1}x_2^{m_2} \cdots x_n^{m_n} \) in \( P \).

3. **Moment problem.** A solution of the moment problem for the set \( E \) is given in the following theorem:

**Theorem.** Given \{ \( U_{(k)}(x) \) \}, \{ \( V_{(k)}(x) \) \}, \{ \( L_v \) \}, and \( T \) satisfying the conditions above, and such that \( TS_v(x, y) = K_v(x, y) \geq 0 \) for all \( (x) \), \( (y) \subseteq E \), and all \( v \), and such that for any \( f \) integrable over \( E \) \( T\mathbb{S}_v(f, x) = f(x) \) for every \( (x) \subseteq E \) for which \( f \) is continuous, and uniformly on \( E \) if \( f \) is continuous on \( E \), then in order that a sequence \( \{ \mu_{(m)} \} \) be expressible in the form

\[
\mu_{m_1,m_2,\ldots,m_n} = \int_E x_1^{m_1}x_2^{m_2} \cdots x_n^{m_n}d\Phi(E),
\]

where \( \Phi \) is completely additive, defined over at least all Borel sets of \( R^n \), and with

(1) \( \int_E d\Phi(E) \leq M \),

(2) \( \Phi(B) \geq 0 \),

(3) \( \Phi(B) = \int_B \phi dx \) and with

(3a) \( \phi \in L^1_B, p > 1 \),

(3b) \( \phi \in L^1_B \),

(3c) \( \phi \leq M \),

(3d) \( \phi \in C_B \),

it is necessary and sufficient that

(1) \( \int_E \mu_{(y)} \left\{ K_v(x, y) \right\} dx \leq M \) for all \( v \),

(2) \( \mu_{(y)} \left\{ K_v(x, y) \right\} \geq 0 \) for all \( (x) \subseteq E \) and all \( v \),

(3a) \( \int_E \mu_{(v)} \left\{ K_v(x, y) \right\} vdx \leq M \) for all \( v \),

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\(^4\) Zygmund, loc. cit., pp. 40–43.
\[ \lim_{r \to -\infty} \int_E \mu(y) \{ K_r(x, y) \} - \mu(y) \{ K_r(x, y) \} \, dx = 0, \]
\[ | \mu(y) \{ K_r(x, y) \} | \leq M \text{ for all } (x) \in E \text{ and all } \nu, \]
\[ \lim_{r \to -\infty} | \mu(y) \{ K_r(x, y) \} - \mu(y) \{ K_r(x, y) \} | = 0 \text{ uniformly in } (x) \in E. \]

The proof in each of the six cases closely parallels that of Hausdorff. The proof is given for case (1) to indicate the modifications:

**Necessity.** We have

\[ | \mu(y) \{ K_r(x, y) \} | = \left| \int_E K_r(x, y) \, d\Phi(E) \right| \]
\[ \leq \int_E K_r(x, y) \, d\Phi(E), \]
\[ \int_E | \mu(y) \{ K_r(x, y) \} | \, dx \leq \int_E \left\{ \int_E K_r(x, y) \, dx \right\} \, d\Phi(E) \]
\[ \leq C \int_E | d\Phi(E) | \leq M \]

for

\[ K_r(x, y) = \sum_{j=0}^{\infty} a_{rj} \sum_{(k) \in L_j} U_{(k)}(x) V_{(k)}(y) \]
\[ \int_E K_r(x, y) \, dx = \sum_{j=0}^{\infty} a_{rj} \sum_{(k) \in L_j} V_{(k)}(y) \int_E U_{(k)}(x) \, dx \]
\[ = \sum_{j=0}^{\infty} a_{rj} \leq \sum_{j=0}^{\infty} | a_{rj} | \leq C. \]

**Sufficiency.** Let

\[ \Phi_r(B) = \int_B \mu(y) \{ K_r(x, y) \} \, dx, \]
\[ \int_E | d\Phi_r(E) | = \int_E | \mu(y) \{ K_r(x, y) \} | \, dx \leq M \]

and, by a well known theorem of Helly, there is a subsequence \{ \Phi_{rj} \} and a function \Phi such that \int_E | d\Phi(E) | \leq M and \Phi_r(B) \to \Phi(B), and also \int_E V_{(k)}(y) d\Phi_r(E) \to \int_E V_{(k)}(y) d\Phi(E) \text{ whence } \mu(y) \{ V_{(k)}(y) \} = \int_E V_{(k)}(y) d\Phi(E), \text{ and } \Phi \text{ is a solution.}

**4. Examples and conclusion.** If \( E \) is the unit sphere in \( \mathbb{R}^n \), \{ \( U_{(k)}(x) \) \} and \{ \( V_{(k)}(x) \) \} may be taken as the normalized polynomials of Appell-
Didon,\(^5\) \((k) \in L_\nu\) to mean \(\sum_{i=1}^n k_i \leq \nu\), and \(T\) any \((C, r)\) with \(r \geq n + 1\).\(^6\) In particular, for \(n = 1\) this reduces to the Hausdorff solution for the unit interval. If \(E\) is the circumference of the unit circle we may set \(U_0(x) = V_0(x) = (2\pi)^{-1/2}\), and, for \(k > 0\),

\[
U_{2k}(x) = V_{2k}(x) = (\pi)^{-1/2} \cos k\theta, \quad U_{2k-1}(x) = V_{2k-1}(x) = (\pi)^{-1/2} \sin k\theta
\]

with \((s) \in L_\nu\) meaning \(s \leq 2\nu\), \(T\) any \((C, r)\) with \(r \geq 1\).\(^7\) Sequences \(\{U_0(x)\}\) and \(\{V_0(x)\}\) can be constructed by the Schmidt process for any bounded region in \(R^n\). It would be interesting to know whether regular Toeplitz transformations of the type required for the present theorem exist in general.

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\(^7\) L. Fejer's theorem. See, for instance, Zygmund, loc. cit., p. 45.