NON-INvolutorial SPACE TRANSFORMATIONS
ASSOCIATED WITH A Q_{1,2} CONGRUENCE

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De Paolis¹ discussed the involutorial transformations associated with the congruence of lines meeting a curve of order \( m \) and an \( (m-1) \)-fold secant, while Vogt² studied the transformation \( T \) for a linear congruence and bundle of lines. In the present paper the transformations associated with the congruence of lines on a conic and a secant of it are discussed.

Given a conic \( r \), a line \( s \) meeting \( r \) once, and two projective pencils of surfaces

\[
|F_{n+m+1}| : r^ns^mg ; \quad |F'_{n'+m'+1}| : r'^ns'^mg',
\]

where \( n \leq m + 1, \ n' \leq m' + 1, \ [r, s] = A \), and \( g, g' \) the residual base curves.

Through a generic point \( P \), there passes a single surface \( F \) of \( |F| \). The unique line \( t \) through \( P, r, s \) meets the associated \( F' \) in one residual point \( P' \), image \((T)\) of \( P \). The transformations to be considered are of three types:

Case I. \( n = m + 1, \ n' = m' + 1 \).
Case II. \( n < m + 1, \ n' < m' + 1 \).
Case III. \( n = m + 1, \ n' < m' + 1 \).

CASE I

Given

\[
|F_{2n}| : r^n s^{n-1} g ; \quad |F'_{2n'}| : r'^n s'^{n'-1} g',
\]

where \( g, g' \) are of order \( n^2 + 2n - 1, \ n'^2 + 2n' - 1 \). The curve \( g \) meets \( r, s \) in \( n^2 + 2n - 1, \ n^2 - 1 \) points respectively.

The conic \( r \) is a fundamental curve whose image \((T^{-1})\) is \( R: r^{n+n'} \), since there are \((n+n')\) invariant directions through each point on \( r \). \( R \) is generated by a monoidal plane curve of order \( n+n'+1 \), one curve on each plane of the pencil \((O, s) = w, \) as \( O \), describes \( r \). The fundamental line \( s \) has for image \((T^{-1})\) a surface \( S: s^{n+n'-1} \), of which \( n+n'-2 \) branches are invariant. \( A \) is a fundamental point of the first kind, whose image \((T^{-1})\) is the plane \( u: r \). In the plane \( v: s \) and tangent

² Vogt, Zentrale und windschiefe Raum-Verwandtschaften, Jahresbericht der Schlesischen Gesellschaft für Vaterländische Kultur, class 84, 1906, pp. 8–16.
to \( r \) there is a curve \( C_{n+r} \), image \((T^{-1})\) of the intersection of \( r, s \) at \( A \), which lies on \( R, S \). The tangent line \([u, v]\) to \( r \) at \( A \) lies on the surface \( R \).

From any point \( Q' \) on \( g' \), there is a unique transversal \( t \) meeting \( r, s \). Any point \( P \) on \( t \) determines an \( F \) and \( t \) meets the associated \( F' \) in a residual point \( Q' \), thus \( Q' \sim (T^{-1})t \). Every point \( P' \) on \( t \) determines the same \( F' \) and \( t \) meets the associated \( F \) in one point \( P \); thus \( P \sim (T)t \).

Considering all points on \( g' \)

\[ g' \sim (T^{-1})G; \quad g_x \sim (T)G, \]

where \( g_x \) is the locus of points \( P \). Similarly

\[ g \sim (T)G'; \quad g'_y \sim (T^{-1})G'. \]

The eliminant of the parameter from \(|F|, |F'|\) is a point-wise invariant surface \( K_{2n+2n'} \). A generic plane meets every line of the pencil \((Au)\), hence the homaloidal surfaces have an additional fixed direction \( d \) through \( A \).

The table of characteristics for \( T^{-1} \) is

\[
\begin{align*}
\pi' &\sim \phi_{2n+2n'+2}; & A^{n+n'+1+d} & r^{n+n'+1} & s^{n+n'} & g & g_x, \\
K &\sim K_{2n+2n'}; & A^{n+n'} & r^{n+n'} & s^{n+n'-2} & g & g_x \quad g' \quad g'_y, \\
r &\sim R_{2n+2n'+1}; & A^{n+n'+d} & r^{n+n'} & s^{n+n'} & g & g_x \quad C_{n+n'} \quad [u, v], \\
s &\sim S_{2n+2n'}; & A^{n+n'} & r^{n+n'} & s^{n+n'-1} & g & g_x \quad C, \\
g' &\sim G_{4n'}; & A^{2n'} & r^{2n'} & s^{2n'} & g' \quad g_x, \\
g'_y &\sim G_{4n}; & A^{2n} & r^{2n} & s^{2n} & g \quad g'_y, \\
A &\sim u: & A & r, \\
J &\equiv u^3RSGG'.
\end{align*}
\]

The intersection of two \( \phi' \)-surfaces gives the order of \( g_y \), \( y = n^2 + 2nn' + 2n + 1 \). The curve \( g_y' \) meets \( r, s \) in \( y, y = 2n \) points respectively.

The equations of \( T^{-1} \) are \( r x_i = R y_i - K z_i = S u y_i + K w_i \), where \( z_i, w_i \) are the points \([t, r], [t, s]\).

**Case II**

Given

\[
|F_{n+m+1}| : r^n s^m g; \quad |F'_{n+m'+1}| : r'^n s'^m' g',
\]

where \( g, g' \) are of order \( 2mn + 2m + 2n - n^2 + 1, 2m'n' + 2m' + 2n' - n'^2 + 1 \). The curve \( g \) meets \( r, s \) in \( 2mn + 4n - n^2, 2mn + 2m - n^2 \) points respectively.
A is a fundamental point of the second kind with image \((T^{-1})C_{n+n'+1}: A^{n+n'}\) in the plane \(v\).

The image \((T^{-1})\) of a point on \(s\) is a curve \(s_{m+m'+2}\) on the quadric cone on \(r\), with a \((m+m')\)-fold point at the vertex and one point on each generator. This curve generates the surface \(S\). The equations of \(T\) are

\[
\tau x = R y_i - K z_i = S y_i + K w_i.
\]

The table of characteristics for \(T^{-1}\) is

\[
\begin{align*}
\pi' &\sim \phi_{n+n'+m+m'+4}: \quad r^{n+n'+1} s^{m+m'+2} \quad g \tilde{g}', \\
K &\sim K_{n+n'+m+m'+2}: \quad r^{n+n'} s^{m+m'} \quad g \tilde{g} \quad g' \tilde{g}', \\
r &\sim R_{n+n'+m+m'+3}: \quad r^{n+n'} s^{m+m'+2} \quad g \tilde{g} \quad C_{n+n'+1}, \\
s &\sim S_{n+n'+m+m'+3}: \quad r^{n+n'+1} s^{m+m'} \quad g \tilde{g} \quad C_{n+n'+1}, \\
g' &\sim G_{2n'+2m'+3}: \quad r^{2n'+1} s^{2m'+2} \quad g' \tilde{g}, \\
g' &\sim G_{2n+2m+3}: \quad r^{2n+1} s^{2m+2} \quad g \tilde{g}', \\
J &\equiv \text{RSGG}',
\end{align*}
\]

where \(y = 2mn + 2m'n + 2mn' + 3m + 3n + m' + n' - n + 5 - 2nn'\). The curve \(\tilde{g}'\) meets \(r, s\) in \([y - (2m - 2n + 1)], [y - (2n + 1)]\) points respectively.

**CASE III**

Given

\[
|F_{2n}| = r^{n} s^{n-1} g, \quad |F'_{m'+m+1}| = r^{n'} s^{m'} g',
\]

where \(g, g'\) are of order \(n^2 + 2n - 1, 2m'n' + 2m' + 2n' - n'^2 + 1\). The curve \(g\) meets \(r, s\) in \(n^2 + 2n - 1, n^2 - 1\) points, and \(g'\) meets \(r, s\) in \(2m'n' + 4n' - n'^2, 2m'n' + 2m' - n'^2\) points respectively.

In \(T^{-1} (T) A\) is a fundamental point of the second (first) kind with image \(C_{n+n'} (u)\). For some point \(D\) on a line \(P'\overline{A}\) of the pencil \((Au)\), the associated \(F\) is the one determined by the direction \(P'\overline{A}\); thus \(D \sim (T^{-1})P'\overline{A}\). The locus of \(D\) is a curve \(\delta_{m'-n'-1}: A^{m'-n'}\) such that \(\delta \sim (T^{-1})u\).

Since \([r, \delta] = (m' - n' + 2)\) points aside from \(A\), \(R: (m' - n' + 2)\) lines of the pencil \((Au)\), hence \(R: A^{n+m'+2}\). The image \((T^{-1})\) of \(A\) as a point on \(s\) is \(C_{n+n'+1}\) and the \((m' - n')\) tangents to \(\delta\) at \(A\), hence \(S: A^{n+m'+1}\).

For the \((2m' - 2n' + 1)\) points, aside from those on \(r, t\), in which \(g'\) meets \(u, t\) becomes a line of the pencil \((Au)\). Therefore \(\tilde{g}'\overline{z}: A^{2m' - 2n' + 1}\) and \([g', \delta] = (2m' - 2n' + 1)\) points.
The table of characteristics for $T^{-1}$ is

\[ \pi' \sim \phi_{2n+m'+m'+3}: \quad A^{n+m'+1} \quad r^{n+n'+1} \quad s^{n+m'+1} \quad g' \quad g_y', \]

\[ K \sim K_{2n+m'+m'+1}: \quad A^{n+m'} \quad r^{n+n'} \quad s^{n+m'-1} \quad g' \quad g_y', \quad \delta, \]

\[ r \sim R_{2n+m'+m'+2}: \quad A^{n+m'} \quad r^{n+n'} \quad s^{n+m'+1} \quad g' \quad g_x \quad C_{n+n'+1}, \]

\[ s \sim S_{2n+m'+m'+2}: \quad A^{n+m'+1} \quad r^{n+n'+1} \quad s^{n+m'} \quad g' \quad g_x \quad C_{n+n'+1}, \]

\[ g' \sim G_{2n'+2m'+3}: \quad A^{2m'+2} \quad r^{2n'+3} \quad s^{2m'+2} \quad g' \quad g_x, \]

\[ \delta \sim u: \quad A \quad r \quad \delta, \]

\[ J = uRSGG', \]

where $y = n^2 + 2m'n + 4n + 1$. The curve $g_y'$ meets $r, s$ in $y, y-2n$ points respectively. The equations of $T^{-1}$ are $\tau x = R_y - K_z = S_y + K w_i$.

The table of characteristics for $T$ is

\[ \pi' \sim \phi_{2n+n'+m'+3}: \quad A^{n+n'+1} \quad r^{n+n'+1} \quad s^{n+m'+1} \quad g' \quad g_y', \]

\[ r \sim R_{2n+n'+m'+2}: \quad A^{n+n'} \quad s^{n+m'+1} \quad g' \quad g_y' \quad C_{n+n'} \quad [u, v] \delta, \]

\[ s \sim S_{2n+n'+m'+1}: \quad A^{n+n'} \quad s^{n+m'} \quad g' \quad g_y' \quad C_{n+n'}, \]

\[ g \sim G_{4n}, \quad g_x \sim G_{2n'+2m'+3}, \]

\[ A \sim u: \quad A r\delta, \quad J' = u'RS'G'G', \]

where $x = 2m'n + 2m'n - n'^2 + 3m' + n' + 2n + 4$. The curve $g_x$ meets $r, s$ in $x = (2m' - 2n' + 1), x = (2m + 2)$ points respectively. The equations of $T$ are $\tau'y = R'x_i + Kz_i = S'ux_i - K w_i$.

In each of the three cases there exists a monoidal transformation in the plane $w$. The space transformations are generated by allowing the vertex to describe the conic $r$, and the plane to generate the pencil on $s$.

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