NON-INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH A $Q_{1,2}$ CONGRUENCE

A. B. CUNNINGHAM

De Paolis[1] discussed the involutorial transformations associated with the congruence of lines meeting a curve of order m and an $(m-1)$-fold secant, while Vogt[2] studied the transformation T for a linear congruence and bundle of lines. In the present paper the transformations associated with the congruence of lines on a conic and a secant of it are discussed.

Given a conic r, a line s meeting r once, and two projective pencils of surfaces

$$|F_{n+m+1}|: \quad r^n s^m g; \qquad |F'_{n'+m'+1}|: \quad r^{n'} s^{m'} g',$$

where $n \leqq m+1$, $n' \leqq m'+1$, $[r, s]=A$, and g, g' the residual base curves.

Through a generic point P, there passes a single surface F of $|F|$. The unique line t through P, r, s meets the associated F' in one residual point P', image (T) of P. The transformations to be considered are of three types:

Case I. $n=m+1$, $n'=m'+1$.
Case II. $n<m+1$, $n'<m'+1$.
Case III. $n=m+1$, $n'<m'+1$.

Case I

Given

$$|F_{2n}|: \quad r^n s^{n-1} g; \qquad |F'_{2n'}|: \quad r^{n'} s^{n'-1} g';$$

where g, g' are of order n^2+2n-1, $n'^2+2n'-1$. The curve g meets r, s in n^2+2n-1, n^2-1 points respectively.

The conic r is a fundamental curve whose image (T^{-1}) is R: $r^{n+n'}$, since there are $(n+n')$ invariant directions through each point on r. R is generated by a monoidal plane curve of order $n+n'+1$, one curve on each plane of the pencil $(O_r s)=w$, as O_r describes r. The fundamental line s has for image (T^{-1}) a surface S: $s^{n+n'-1}$, of which $n+n'-2$ branches are invariant. A is a fundamental point of the first kind, whose image (T^{-1}) is the plane $u: r$. In the plane $v: s$ and tangent

[1] De Paolis, *Alcuni particolari trasformazioni involutori dello spazio*, Rendiconti dell' Accademia dei Lincei, Rome, (4), vol. 1 (1885), pp. 735–742, 754–758.

[2] Vogt, *Zentrale und windschiefe Raum-Verwandtschaften*, Jahresbericht der Schlesischen Gesellschaft für Vaterländische Kultur, class 84, 1906, pp. 8–16.

to r there is a curve $C_{n+n'}$, image (T^{-1}) of the intersection of r, s at A, which lies on R, S. The tangent line $[u, v]$ to r at A lies on the surface R.

From any point Q' on g', there is a unique transversal t meeting r, s. Any point P on t determines an F and t meets the associated F' in a residual point Q', thus $Q' \sim (T^{-1})t$. Every point P' on t determines the same F' and t meets the associated F in one point $\overline{P}$; thus $\overline{P} \sim (T)t$. Considering all points on g'

$$g' \sim (T^{-1})G; \qquad \bar{g}_x \sim (T)G,$$

where $\bar{g}_x$ is the locus of points $\overline{P}$. Similarly

$$g \sim (T)G'; \qquad \bar{g}_y' \sim (T^{-1})G'.$$

The eliminant of the parameter from $|F|$, $|F'|$ is a point-wise invariant surface $K_{2n+2n'}$. A generic plane meets every line of the pencil (Au), hence the homaloidal surfaces have an additional fixed direction d through A.

The table of characteristics for T^{-1} is

$\pi' \sim$	$\phi_{2n+2n'+2}$:	$A^{n+n'+1+d}$	$r^{n+n'+1}$	$s^{n+n'}$	g	$\bar{g}_x$,		
$K \sim$	$K_{2n+2n'}$:	$A^{n+n'}$	$r^{n+n'}$	$s^{n+n'-2}$	g	$\bar{g}_x$	g'	$\bar{g}_y'$,
$r \sim$	$R_{2n+2n'+1}$:	$A^{n+n'+d}$	$r^{n+n'}$	$s^{n+n'}$	g	$\bar{g}_x$	$C_{n+n'}$	$[u, v]$,
$s \sim$	$S_{2n+2n'}$:	$A^{n+n'}$	$r^{n+n'}$	$s^{n+n'-1}$	g	$\bar{g}_x$	C,	
$g' \sim$	$G_{4n'}$:	$A^{2n'}$	$r^{2n'}$	$s^{2n'}$	g'	$\bar{g}_x$,		
$\bar{g}_y' \sim$	G_{4n}:	A^{2n}	r^{2n}	s^{2n}	g	$\bar{g}_y'$,		
$A \sim$	u:	A	r,					

$J \equiv u^3RSGG'$.

The intersection of two ϕ'-surfaces gives the order of $\bar{g}_y'$, $y = n^2 + 2nn' + 2n + 1$. The curve $\bar{g}_y'$ meets r, s in y, $y - 2n$ points respectively.

The equations of T^{-1} are $\tau x_i = Ry_i - Kz_i = Suy_i + Kw_i$, where z_i, w_i are the points $[t, r]$, $[t, s]$.

Case II

Given

$$|F_{n+m+1}| : \quad r^n s^m g; \qquad |F'_{n'+m'+1}| : \quad r^{n'} s^{m'} g',$$

where g, g' are of order $2mn + 2m + 2n - n^2 + 1$, $2m'n' + 2m' + 2n' - n'^2 + 1$. The curve g meets r, s in $2mn + 4n - n^2$, $2mn + 2m - n^2$ points respectively.

A is a fundamental point of the second kind with image $(T^{-1})C_{n+n'+1}$: $A^{n+n'}$ in the plane v.

The image (T^{-1}) of a point on s is a curve $s_{m+m'+2}$ on the quadric cone on r, with a $(m+m')$-fold point at the vertex and one point on each generator. This curve generates the surface S. The equations of T are

$$\tau x = Ry_i - Kz_i = Sy_i + Kw_i.$$

The table of characteristics for T^{-1} is

$$\begin{array}{llllllll}
\pi' \sim & \phi_{n+n'+m+m'+4}: & r^{n+n'+1} & s^{m+m'+2} & g & \bar{g}_x, & & \\
K \sim & K_{n+n'+m+m'+2}: & r^{n+n'} & s^{m+m'} & g & \bar{g}_x & g' & \bar{g}'_y, \\
r \sim & R_{n+n'+m+m'+3}: & r^{n+n'} & s^{m+m'+2} & g & \bar{g}_x & C_{n+n'+1}, & \\
s \sim & S_{n+n'+m+m'+3}: & r^{n+n'+1} & s^{m+m'+1} & g & \bar{g}_x & C_{n+n'+1}, & \\
g' \sim & G_{2n'+2m'+3}: & r^{2n'+1} & s^{2m'+2} & g' & \bar{g}_x, & & \\
\bar{g}'_y \sim & G'_{2n+2m+3}: & r^{2n+1} & s^{2m+2} & g & \bar{g}'_y, & & \\
J \equiv RSGG', & & & & & & &
\end{array}$$

where $y = 2mn + 2m'n + 2mn' + 3m + 3n + m' + n' - n + 5 - 2nn'$. The curve $\bar{g}'_y$ meets r, s in $[y-(2m-2n+1)]$, $[y-(2n+1)]$ points respectively.

Case III

Given

$$|F_{2n}| : \; r^n s^{n-1} g; \qquad |F'_{n'+m'+1}| : \; r^{n'} s^{m'} g',$$

where g, g' are of order n^2+2n-1, $2m'n'+2m'+2n'-n'^2+1$. The curve g meets r, s in n^2+2n-1, n^2-1 points, and g' meets r, s in $2m'n'+4n'-n'^2$, $2m'n'+2m'-n'^2$ points respectively.

In T^{-1} (T) A is a fundamental point of the second (first) kind with image $C'_{n+n'}$ (u). For some point D on a line $\overline{P'A}$ of the pencil (Au), the associated F is the one determined by the direction $\overline{P'A}$; thus $D \sim (T^{-1})\overline{P'A}$. The locus of D is a curve $\delta_{m'-n'+1}$: $A^{m'-n'}$ such that $\delta \sim (T^{-1})u$.

Since $[r, \delta] = (m'-n'+2)$ points aside from A, R: $(m'-n'+2)$ lines of the pencil (Au), hence R: $A^{n+m'+2}$. The image (T^{-1}) of A as a point on s is $C_{n+n'+1}$ and the $(m'-n')$ tangents to δ at A, hence S: $A^{n+m'+1}$.

For the $(2m'-2n'+1)$ points, aside from those on r, in which g' meets u, t becomes a line of the pencil (Au). Therefore $\bar{g}_x$: $A^{2m'-2n'+1}$ and $[g', \delta] = (2m'-2n'+1)$ points.

The table of characteristics for T^{-1} is

$$
\begin{array}{rlllllll}
\pi' \sim \phi_{2n+n'+m'+3}: & A^{n+m'+1} & r^{n+n'+1} & s^{n+m'+1} & g & \bar{g}_x, & & \\
K \sim K_{2n+n'+m'+1}: & A^{n+m'} & r^{n+n'} & s^{n+m'-1} & g & \bar{g}_x & g' & \bar{g}'_y\,\delta, \\
r \sim R_{2n+n'+m'+2}: & A^{n+m'+2} & r^{n+n'} & s^{n+m'+1} & g & \bar{g}_x & C_{n+n'+1}, & \\
s \sim S_{2n+n'+m'+2}: & A^{n+m'+1} & r^{n+n'+1} & s^{n+m'} & g & \bar{g}_x & C_{n+n'+1}, & \\
g' \sim G_{2n'+2m'+3}: & A^{2m'+2} & r^{2n'+1} & s^{2m'+2} & g' & \bar{g}_x, & & \\
\bar{g}_y \sim G'_{4n}: & A^{2n} & r^{2n} & s^{2n} & g & \bar{g}'_y, & & \\
\delta \sim u: & A & r & \delta, & & & & \\
J \equiv uRSGG', & & & & & & &
\end{array}
$$

where $y=n^2+2m'n+4n+1$. The curve $\bar{g}_y'$ meets r, s in y, $y-2n$ points respectively. The equations of T^{-1} are $\tau x=Ry_i-Kz_i=Sy_i+Kw_i$.

The table of characteristics for T is

$$
\begin{array}{rllllll}
\pi \sim \phi'_{2n+n'+m'+3}: & r^{n+n'+1} & s^{n+m'+1} & g' & \bar{g}'_y & \delta, & \\
r \sim R_{2n+n'+m'+2}: & r^{n+n'} & s^{n+m'+1} & g' & \bar{g}'_y & C'_{n+n'} & [u,\, v]\delta, \\
s \sim S'_{2n+n'+m'+1}: & r^{n+n'} & s^{n+m'} & g' & \bar{g}'_y & C'_{n+n'}, & \\
g \sim G'_{4n}, & \bar{g}_x \sim G_{2n'+2m'+3}, & & & & & \\
A \sim u: \quad Ar\delta, & J' \equiv u^2R'S'G'G, & & & & &
\end{array}
$$

where $x=2m'n'+2m'n-n'^2+3m'+n'+2n+4$. The curve $\bar{g}_x$ meets r, s in $x-(2m'-2n'+1)$, $x-(2m+2)$ points respectively. The equations of T are $\tau'y=R'x_i+Kz_i'=S'ux_i-Kw_i'$.

In each of the three cases there exists a monoidal transformation in the plane w. The space transformations are generated by allowing the vertex to describe the conic r, and the plane to generate the pencil on s.

West Virginia University