ON BIORTHOGONAL MATRICES[1]

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Introduction. Consider the basis consisting of a number system $\mathfrak{A}$ of type D, two general ranges $\mathfrak{P}^1$, $\mathfrak{P}^2$, and two positive hermitian matrices ϵ^1, ϵ^2. We introduce two binary relations for pairs of non-modular matrices. The matrices κ^{12}, ϕ^{21} are said to be contraceding as to $\epsilon^1\epsilon^2_\kappa$, ϵ^2 in case κ^{12}, ϕ^{21} are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively and such that $J^{2\kappa}\kappa^{12}\mu^2 = J^2\phi^{*12}\mu^2$ for every μ^2 in the set $\mathfrak{M}(\epsilon^2_\kappa \cap \epsilon^2)$. It is evident that when κ^{12} is of type $\mathfrak{M}(\epsilon^1)\overline{\mathfrak{M}}(\epsilon^2)$, then the contracedence property implies that $J^2\kappa^{12}\phi^{21} = \epsilon^1_\kappa$ but not conversely. The main results are stated in Theorems 2 and 3. We next consider ϵ^1_0, ϵ^1_1 both idempotent as to ϵ^1. Suppose that κ^{12} is by columns of $\mathfrak{M}(\epsilon^1_0)$ and ϕ^{21} is by rows-conjugate of $\mathfrak{M}(\epsilon^1_1)$. Take any pair of vectors μ^1, ν^1 modular as to ϵ^1_0, ϵ^1_1 respectively such that $J^1\kappa^{*21}\mu^1$, $J^1\phi^{21}\nu^1$ are in $\mathfrak{M}(\epsilon^2)$. If $J^1\bar{\mu}^1\nu^1$ is equal to the inner product $J^2(J^1\bar{\mu}^1\kappa^{12}, J^1\phi^{21}\nu^1)$, then κ^{12}, ϕ^{21} are said to be biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$. When $\epsilon^1_1 = \epsilon^1_0$, then κ^{12}, ϕ^{21} are said to be biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^2$ in case they are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_0\epsilon^2$. With proper restrictions imposed upon κ^{12}, ϕ^{21}, we obtain the contracedence property. In a later paper, we shall establish the relations of biorthogonality and a certain mode of interchange of integration processes.

1. **Preliminary results.** Consider the basis $\mathfrak{A}$, $\mathfrak{P}^1$, $\mathfrak{P}^2$, ϵ^1, and κ^{12} which is by columns of $\mathfrak{M}(\epsilon^1)$. E. H. Moore's generalized Fourier processes give $\epsilon^2_\kappa \equiv J^1\kappa^{*21}\kappa^{12}$ and $\epsilon^1_\kappa \equiv J^{2\kappa}\kappa^{12}\kappa^{*21}$. The spaces $\mathfrak{M}(\epsilon^1_\kappa)$ and $\mathfrak{M}(\epsilon^2_\kappa)$ are in one-to-one correspondence (denoted by $\leftrightarrow$) via the transformations $J^1\kappa^{*21}$ and $J^{2\kappa}\kappa^{12}$, and the correspondences are orthogonal in the sense that the moduli of the corresponding vectors are preserved.[2]

(A)[3] Suppose that $\mathfrak{M}_1(\epsilon^1_\kappa) \leftrightarrow \mathfrak{M}_1(\epsilon^2_\kappa)$ and $\mathfrak{M}_2(\epsilon^1_\kappa) \leftrightarrow \mathfrak{M}_2(\epsilon^2_\kappa)$ via the transformations $J^1\kappa^{*21}$, $J^{2\kappa}\kappa^{12}$. Then $\mathfrak{M}_1(\epsilon^1_\kappa)$ is a subset of $\mathfrak{M}_2(\epsilon^1_\kappa)$ if and only if $\mathfrak{M}_1(\epsilon^2_\kappa)$ is a subset of $\mathfrak{M}_2(\epsilon^2_\kappa)$; $\mathfrak{M}_1(\epsilon^1_\kappa)$ is linearly J^1-closed if and only if $\mathfrak{M}_1(\epsilon^2_\kappa)$ is linearly $J^{2\kappa}$-closed; and $\mathfrak{M}_1(\epsilon^1_\kappa)$ is everywhere dense in $\mathfrak{M}_2(\epsilon^1_\kappa)$ if and only if $\mathfrak{M}_1(\epsilon^2_\kappa)$ is everywhere dense in $\mathfrak{M}_2(\epsilon^2_\kappa)$.

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[2] For a concise outline of Moore's generalized Fourier theory and its related topics, *see* Moore, *General Analysis*, I, pp. 19–26. For an important classical instance, *see* E. Schmidt, *Über die Auflösung linearer Gleichungen mit unendlichvielen Unbekannten*, Rendiconti del Circolo Matematico di Palermo, vol. 25 (1908), pp. 56–77.

[3] For the demonstrations of the following results, *see* the author's forthcoming paper *On non-modular matrices*.

Let ϵ^2 be a positive hermitian matrix. (B) The class of all vectors μ^1 modular as to ϵ^1 such that $J^1\kappa^{*21}\mu^1$ is in $\mathfrak{M}(\epsilon^2)$ will be denoted by $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$. The intersection of $\mathfrak{M}(\epsilon^2)$ and $\mathfrak{M}(\epsilon_\kappa^2)$ will be denoted by $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$. (C) The set $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$ is identical with the class of vectors $J^1\kappa^{*21}\mu^1$ for all μ^1 in $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$. Moreover, (D) the sets $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$ are in one-to-one (orthogonal) correspondence via the transformations $J^1\kappa^{*21}$ and $J^{2\kappa}\kappa^{12}$. (E) The set $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$ is the linear extension of the sum of $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and the orthogonal complement within $\mathfrak{M}(\epsilon^1)$ of $\mathfrak{M}(\epsilon_\kappa^1)$. (F) The class $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$ is everywhere dense in $\mathfrak{M}(\epsilon^1)$ if and only if $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$ is everywhere dense in $\mathfrak{M}(\epsilon_\kappa^2)$.

2. **Contraceding pairs of matrices.** The basis of the paper consists of a number system $\mathfrak{A}$ of type D, two general ranges $\mathfrak{P}^1$, $\mathfrak{P}^2$, and two positive hermitian matrices ϵ^1, ϵ^2.

LEMMA 1. *Suppose that κ^{12} is by columns of $\mathfrak{M}(\epsilon^1)$. Let $\mathfrak{M}_0(\epsilon^1)$ be a subset of $\mathfrak{M}(\epsilon^1)$; then $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ is contained in (contains) $\mathfrak{M}_0(\epsilon^1)$ only if $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$ is contained in (contains) the class of vectors $J^1\kappa^{*21}\mu^1$ for all μ^1 in $\mathfrak{M}_0(\epsilon^1)$. The converses are valid provided that $\mathfrak{M}_0(\epsilon^1)\subset\mathfrak{M}(\epsilon_\kappa^1)$.*

PROOF. The lemma follows from (D) and (A) in §1.

If ϕ^{21} is by columns of $\mathfrak{M}(\epsilon^2)$, we introduce the notations due to E. H. Moore:

$$\overset{1}{\epsilon}_{\phi} \equiv \overset{2}{J}\,\overset{*12}{\phi}\,\overset{21}{\phi}, \qquad \overset{2}{\epsilon}_{\phi} \equiv \overset{1\phi}{J}\,\overset{21}{\phi}\,\overset{*12}{\phi}.$$

LEMMA 2. *Suppose that κ^{12}, ϕ^{21} are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively. Then $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ is contained in (contains) $\mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_\phi^1)$ if and only if $\mathfrak{M}(\epsilon_\kappa^2\cap\epsilon^2)$ is contained in (contains) the class of vectors $J^1\kappa^{*21}\mu^1$ for all μ^1 in $\mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_\phi^1)$.*

PROOF. This lemma is a special instance of Lemma 1.

DEFINITION 1. *The matrices κ^{12}, ϕ^{21} are said to be contraceding as to $\epsilon^1\epsilon_\kappa^2\epsilon^2$ in case κ^{12}, ϕ^{21} are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively and such that $J^{2\kappa}\kappa^{12}\mu^2 = J^2\phi^{*12}\mu^2$ for every μ^2 in the set $\mathfrak{M}(\epsilon_\kappa^2\cap\epsilon^2)$.*

THEOREM 1. *Suppose that κ^{12}, ϕ^{21} are contraceding as to $\epsilon^1\epsilon_\kappa^2\epsilon^2$. Then*
(1) $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2) = [J^2\phi^{*12}\mu^2 \,|\, \mu^2 \text{ in } \mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)] \subset \mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_\phi^1)$;
(2) $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)\subset\mathfrak{M}(\epsilon^2\phi\epsilon_\kappa^1)$;
(3) *$\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)$ if and only if every μ^1 for which $J^1\epsilon_\kappa^1\mu^1 = 0^1$ is in $\mathfrak{M}(\epsilon_\phi^1)$.*

PROOF. Part (1) is a consequence of the fact that

$$\mathfrak{M}(\overset{1}{\epsilon}_{\kappa}\overset{*}{\kappa}\overset{2}{\epsilon}) = [\overset{2\kappa}{J}\,\overset{12}{\kappa}\,\overset{2}{\mu} \,|\, \overset{2}{\mu} \text{ in } \mathfrak{M}(\overset{2}{\epsilon}\cap\overset{2}{\epsilon}_{\kappa})].$$

By hypothesis, we may replace $J^{2\kappa}\kappa^{12}$ by $J^2\phi^{*12}$. Since ϕ^{*12} is modular as to $\epsilon_\phi^1\epsilon^2$ and $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ is a subset of $\mathfrak{M}(\epsilon_\kappa^1)$, every vector of the form $J^2\phi^{*12}\mu^2$ must belong to $\mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_\phi^1)$. This proves (1). Part (2) is obvious from (1). Part (3) follows from (1) and (E).

THEOREM 2. *Suppose that* ϕ^{21}, κ^{12} *are contraceding as to* $\epsilon^2\epsilon_\phi^1\epsilon^2$. *Then*
(1) $O^1\mathfrak{M}(\epsilon_\kappa^1)\subset\mathfrak{M}(\epsilon_\phi^1)$ *if and only if* κ^{12} *is complete by columns of* $\mathfrak{M}(\epsilon^1)$;
(2) *the following four assertions are equivalent*: (i) $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon_\phi^1)$; (ii) κ^{12} *is complete by columns of* $\mathfrak{M}(\epsilon^1)$ *and* $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)\subset\mathfrak{M}(\epsilon_\phi^2\phi^*\epsilon^1)$; (iii) κ^{12} *is complete by columns of* $\mathfrak{M}(\epsilon^1)$ *and* κ^{12}, ϕ^{21} *are contraceding as to* $\epsilon^1\epsilon_\kappa^2\epsilon^2$; (iv) $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset[J^2\phi^{*12}\mu^2\,|\,\mu^2$ *in* $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)]$.
If one of the four conditions in (2) *is valid, then* $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)=\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)$ *and* $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)=\mathfrak{M}(\epsilon_\phi^2\phi^*\epsilon^1)$.

PROOF. If κ^{12} is complete by columns of $\mathfrak{M}(\epsilon^1)$, then $O^1\mathfrak{M}(\epsilon_\kappa^1)=[0^1]$, which is, of course, contained in $\mathfrak{M}(\epsilon_\phi^1)$. Conversely, every μ^1 satisfying $J^1\bar{\mu}^1\kappa^{12}=0^2$ is in the orthogonal complement within $\mathfrak{M}(\epsilon^1)$ of $\mathfrak{M}(\epsilon_\kappa^1)$, and hence is in $\mathfrak{M}(\epsilon_\phi^1)$. Thus $J^{1\phi}\phi^{21}\mu^1=0^2$. Since ϕ^{21} is complete by rows-conjugate of $\mathfrak{M}(\epsilon_\phi^1)$, we have $\mu^1=0^1$, proving that κ^{12} is complete by columns of $\mathfrak{M}(\epsilon^1)$.

To prove (i) implies (ii), we observe by (E) that $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$ is the linear extension of the sum of $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and $O^1\mathfrak{M}(\epsilon_\kappa^1)$. If (i) is valid, then by part (1) just proved, κ^{12} is complete by columns of $\mathfrak{M}(\epsilon^1)$. By Lemma 2, the condition that $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon_\phi^1)$ is equivalent to

$$\text{(a)}\qquad \mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)\subset[J^1\kappa^{*21}\mu^1\,|\,\mu^1\text{ in }\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)].$$

By hypothesis, we may replace $J^1\kappa^{*21}$ by $J^{1\phi}\phi^{21}$. Since, by (D),

$$\text{(b)}\qquad [J^{1\phi}\phi^{21}\mu^1\,|\,\mu^1\text{ in }\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)]=\mathfrak{M}(\epsilon_\phi^2\phi^*\epsilon^1),$$

we have (i)→(ii). To prove (ii)→(iii), consider any vector μ^2 in $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$. If (ii) is valid, then, by (D) (cf. (b) above), there exists a vector μ^1 in $\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)$ such that

$$\text{(c)}\qquad \mu^2=J^{1\phi}\phi^{21}\mu^1,$$

whence

$$J^2\phi^{*12}\mu^2=J^2\phi^{*12}J^{1\phi}\phi^{21}\mu^1=J^{1\phi}\epsilon_\phi\mu^1=\mu^1.$$

Now by hypothesis and (c), we have $\mu^2=J^1\kappa^{*21}\mu^1$. As $\epsilon_\kappa^1=\epsilon^1$, it follows that

$$J^{2\kappa}\kappa^{12}\mu^2=J^{2\kappa}\kappa^{12}J^1\kappa^{*21}\mu^1=J^1\epsilon_\kappa\mu^1=\mu^1,$$

proving the condition (iii). By Theorem 1, we have (iii)$\rightarrow$(iv). Since ϕ^{*12} is modular as to $\epsilon^1_\phi\epsilon^2$, we secure (iv)$\rightarrow$(i).

The final statement follows by Theorem 1.

THEOREM 3. *The matrices ϕ^{21}, κ^{12} are contraceding as to $\epsilon^2\epsilon^1_\phi\epsilon^1$, ϕ^{21} is complete by columns of $\mathfrak{M}(\epsilon^2)$, and $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon^1_\phi)$ if and only if κ^{12}, ϕ^{21} are contraceding as to $\epsilon^1\epsilon^2_\kappa\epsilon^2$, κ^{12} is complete by columns of $\mathfrak{M}(\epsilon^1)$, and $\mathfrak{M}(\epsilon^2\phi^*\epsilon^1)\subset\mathfrak{M}(\epsilon^2_\kappa)$.*

PROOF. Apply Theorems 1 and 2.

3. **Biorthogonal matrices.** By considering $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ and $\mathfrak{M}(\epsilon^2_\kappa\cap\epsilon^2)$ as subsets of $\mathfrak{M}(\epsilon^1_\kappa)$ and $\mathfrak{M}(\epsilon^2_\kappa)$ respectively, we have established the one-to-one correspondence between those two subsets. The correspondences are orthogonal. But the Fourier coefficient function of every vector in $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ is also modular as to ϵ^2. This property gives rise to another direction of studying the correspondences between the aforementioned subsets.

DEFINITION 2. *Suppose that ϵ^1_0, ϵ^1_1 are idempotent[4] as to ϵ^1, κ^{12} is by columns of $\mathfrak{M}(\epsilon^1_0)$, and ϕ^{21} is by rows-conjugate of $\mathfrak{M}(\epsilon^1_1)$. Then κ^{12}, ϕ^{21} are said to be biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$ in case*

$$J^1\bar{\mu}^1\nu^1 = J^2(J^1\bar{\mu}^1\kappa^{12}, J^1\phi^{21}\nu^1)$$

for every pair of vectors μ^1, ν^1 in $\mathfrak{M}(\epsilon^1_0\kappa^*\epsilon^2)$, $\mathfrak{M}(\epsilon^1_1\phi\epsilon^2)$ respectively.

It is obvious that κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$ if and only if ϕ^{*12}, κ^{*21} are biorthogonal as to $\epsilon^1\epsilon^1_1\epsilon^1_0\epsilon^2$.

DEFINITION 3. *Suppose that ϵ^1_0 is idempotent as to ϵ^1 and κ^{12}, ϕ^{*12} are by columns of $\mathfrak{M}(\epsilon^1_0)$. Then κ^{12}, ϕ^{21} are said to be biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^2$ if they are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_0\epsilon^2$.*

THEOREM 4. *Suppose that ϵ^1_0, ϵ^1_1 are idempotent as to ϵ^1, and κ^{12}, ϕ^{*12} are by columns of $\mathfrak{M}(\epsilon^1_0)$, $\mathfrak{M}(\epsilon^1_1)$ respectively. Then*

(i) *κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^1_1\epsilon^2$ if κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$, and only if κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^1_{\phi*}\epsilon^2$;*

(ii) *κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^1_1\epsilon^2$ if and only if*

$$J^2(J^1\bar{\mu}^1\phi^{*12}, \mu^2) = J^{2\kappa}(J^1\bar{\mu}^1\kappa^{12}, \mu^2) \tag{4.1}$$

holds for every μ^1 in $\mathfrak{M}(\epsilon^1_1\phi\epsilon^2)$ and every μ^2 in $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$.

(iii) *κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon_{\phi*}\epsilon^2$ if and only if*

[4] ϵ^1_0 is idempotent as to ϵ^1 in case ϵ^1_0 is by columns of $\mathfrak{M}(\epsilon^1)$ and $J^1\epsilon^{1*}_0\epsilon^1_0=\epsilon_0$. See G.A., I, pp. 23–24.

(4.2) $$J^2\bar{\xi}^2\eta^2 = J^1(J^{2\phi^*}\bar{\xi}^2\phi^{21}, J^{2\kappa}\kappa^{12}\eta^2)$$

holds for every ξ^2 *in* $\mathfrak{M}(\epsilon^2\cap\epsilon^2_{\phi^*})$ *and every* η^2 *in* $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$.

PROOF. Since $\mathfrak{M}(\epsilon^1_\kappa)$ and $\mathfrak{M}(\epsilon^1_{\phi^*})$ are subsets of $\mathfrak{M}(\epsilon^1_0)$ and $\mathfrak{M}(\epsilon^1_1)$ respectively, we have the conclusion (i).

To prove the necessity in (ii), consider any μ^2 in the set $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$ and any vector μ^1 in $\mathfrak{M}(\epsilon^1_1\phi\epsilon^2)$. Theorem (D) shows the existence of a vector ν^2 in $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ such that

(a) $$\nu^1 = J^{2\kappa}\kappa^{12}\mu^2, \qquad \mu^2 = J^1\kappa^{*21}\nu^1.$$

By using the fact that κ^{12} is modular as to $\epsilon^1\epsilon^2_\kappa$, we have

(b) $$J^{2\kappa}(J^1\bar{\mu}^1\kappa^{12}, \mu^2) = J^1\bar{\mu}^1\nu^1.$$

Equations (a) and (b) show that the condition is necessary. The sufficiency can be proved similarly.

Now observe that κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^1_{\phi^*}\epsilon^2$ if and only if

(c) $$J^2(J^1\bar{\mu}^1\phi^{*12}, \eta^2) = J^1(\bar{\mu}^1, J^{2\kappa}\kappa^{12}\eta^2)$$

for every μ^1 in $\mathfrak{M}(\epsilon^1_{\phi^*}\phi\epsilon^2)$ and every η^2 in $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$. The latter condition is obviously satisfied if and only if (4.2) holds, since $\mathfrak{M}(\epsilon^1_{\phi^*}\phi\epsilon^2)$ and $\mathfrak{M}(\epsilon^2_{\phi^*}\cap\epsilon^2)$ are in one-to-one correspondence via the transformations $J^1\phi^{21}$ and $J^{2\phi^*}\phi^{*12}$ by (D).

THEOREM 5. *Suppose that* ϵ^1_0, ϵ^1_1, κ^{12}, ϕ^{21} *have the same properties as in the hypothesis of Theorem* 4, *and* $\mathfrak{M}(\epsilon^1_\kappa)$ *contains either* $\mathfrak{M}(\epsilon^1_1)$ *or* $\mathfrak{M}(\epsilon^1_0)$. *Then* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$ *if and only if* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon^1_\kappa\epsilon^1_1\epsilon^2$.

PROOF. The condition is necessary by Theorem 4. If $\mathfrak{M}(\epsilon^1_\kappa)$ contains $\mathfrak{M}(\epsilon^1_0)$, the condition is obviously sufficient. If $\mathfrak{M}(\epsilon^1_\kappa)$ contains $\mathfrak{M}(\epsilon^1_1)$, the sufficiency is proved by the following argument: Let μ^1, ν^1 be vectors in $\mathfrak{M}(\epsilon^1_1\phi\epsilon^2)$, $\mathfrak{M}(\epsilon^1_0\kappa^*\epsilon^2)$ respectively. Since μ^1 is modular as to ϵ^1_κ by hypothesis, we have

$$J^1\bar{\mu}^1\nu^1 = J^1(J^1\bar{\mu}^1\epsilon^1_\kappa, \nu^1) = J^1(\bar{\mu}^1, J^1\epsilon^1_\kappa\nu^1).$$

Also

$$J^1\kappa^{*21}\nu^1 = J^1J^1\kappa^{*21}\epsilon^1_\kappa\nu^1 = J^1(\kappa^{*21}, J^1\epsilon^1_\kappa\nu^1).$$

The vector $J^1\epsilon^1_\kappa\nu^1$ is evidently in the set $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$.

COROLLARY 6. *Suppose that* κ^{12}, ϕ^{*12} *are by columns of* $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^1_\kappa)$

respectively. Then κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_\kappa^1\epsilon^2$ *if and only if* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon_\kappa^1\epsilon^2$.

THEOREM 7. *Suppose that* ϵ_0^1, ϵ_1^1, κ^{12}, ϕ^{21} *have the same properties as in the hypothesis of Theorem* 4, *and* $\mathfrak{M}(\epsilon_\kappa^2 \cap \epsilon^2)$ *is everywhere dense in* $\mathfrak{M}(\epsilon_\kappa^2)$. *Then* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_1^1\epsilon^2$ *and the orthogonal complement within* $\mathfrak{M}(\epsilon_1^1)$ *of* $\mathfrak{M}(\epsilon_{\phi*}^1)$ *is a subset of* $\mathfrak{M}(\epsilon_0^1)$ *if and only if* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_{\phi*}^1\epsilon^2$ *and* ϕ^{21} *is complete by rows-conjugate of* $\mathfrak{M}(\epsilon_1^1)$.

PROOF. The condition is obviously sufficient. For the necessity, we need to show only that ϕ^{21} is complete by rows-conjugate of $\mathfrak{M}(\epsilon_1^1)$. Consider any ξ^1 modular as to ϵ_1^1 such that $J^{11}\phi^{21}\xi^1 = J^1\phi^{21}\xi^1 = 0^2$. Then ξ^1 belongs to $\mathfrak{M}(\epsilon_1^1\phi\epsilon^2)$. Hence

$$J^1\bar{\mu}^1\xi^1 = J^2(J^1\bar{\mu}^1\kappa^{12}, J^1\phi^{21}\xi^1) = 0$$

for every μ^1 belonging to $\mathfrak{M}(\epsilon_0^1\kappa^*\epsilon^2)$. Now if $\mathfrak{M}(\epsilon_\kappa^2 \cap \epsilon^2)$ is everywhere dense in $\mathfrak{M}(\epsilon_\kappa^2)$, then $\mathfrak{M}(\epsilon_0^1\kappa^*\epsilon^2)$ is everywhere dense in $\mathfrak{M}(\epsilon_0^1)$, or, $O^1\mathfrak{M}(\epsilon_0^1\kappa^*\epsilon^2) = O^1\mathfrak{M}(\epsilon_0^1)$. Since ξ^1 is in $\mathfrak{M}(\epsilon_0^1)$ and also in $O^1\mathfrak{M}(\epsilon_0^1)$, it follows that $\xi^1 = 0^1$.

THEOREM 8. *Assume that* κ^{12} *is of type* $\mathfrak{M}(\epsilon^1)\overline{\mathfrak{M}}(\epsilon^2)$, *and* ϕ^{21} *is by rows-conjugate of* $\mathfrak{M}(\epsilon^1)$ *such that* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon_\kappa^1\epsilon_{\phi*}^1\epsilon^2$. *Then*

(1) ϕ^{*12}, κ^{*21} *are contraceding as to* $\epsilon^1\epsilon_{\phi*}^2\epsilon^2$;

(2) $\mathfrak{M}(\epsilon^2 \cap \epsilon_{\phi*}^2) \subset \mathfrak{M}(\epsilon^2\kappa\epsilon_\kappa^1)$;

(3) $\mu^1 = J^2\kappa^{12}J^1\phi^{21}\mu^1$ *for every* μ^1 *in* $\mathfrak{M}(\epsilon_{\phi*}^1\phi\epsilon^2)$;

(4) $\mathfrak{M}(\epsilon^2 \cap \epsilon_{\phi*}^2) \subset [J^1\phi^{21}\mu^1 \mid \mu^1$ *in* $\mathfrak{M}(\epsilon_\kappa^1 \cap \epsilon_{\kappa*}^1)]$;

(5) $\mathfrak{M}(\epsilon_\kappa^1\phi\epsilon^2) \supset \mathfrak{M}(\epsilon_\kappa^1 \cap \epsilon_{\kappa*}^1)$ *if and only if* $\mathfrak{M}(\epsilon^2\kappa\epsilon_\kappa^1)$ *contains* $[J^1\phi^{21}\mu^1 \mid \mu^1$ *in* $\mathfrak{M}(\epsilon_\kappa^1 \cap \epsilon_{\kappa*}^1)]$.

PROOF. Since κ^{12} is by rows-conjugate of $\mathfrak{M}(\epsilon^2 \cap \epsilon_\kappa^2)$, we make use of (4.2) in Theorem 4 and secure

$$J^2\kappa^{12}\mu^2 = J^1\epsilon_\kappa^1 J^{2\phi*}\phi^{*12}\mu^2 = J^{2\phi*}\phi^{*12}\mu^2$$

for every μ^2 belonging to $\mathfrak{M}(\epsilon^2 \cap \epsilon_{\phi*}^2)$. This proves (1). Part (2) is an immediate consequence of (1) just proved. For the demonstration of (3), we note that ϵ_κ^1 is by columns of $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and hence by Definition 2, we have

$$J^1\epsilon_\kappa^1\eta^1 = J^2(J^1\epsilon_\kappa^1\kappa^{12}, J^1\phi^{21}\eta^1) = J^2\kappa^{12}J^1\phi^{21}\eta^1$$

for every η^1 in $\mathfrak{M}(\epsilon_{\phi*}^1\phi\epsilon^2)$. Since $\mathfrak{M}(\epsilon_{\phi*}^1)$ is a subset of $\mathfrak{M}(\epsilon_\kappa^1)$, we have $J^1\epsilon_\kappa^1\eta^1 = \eta^1$, and hence, part (3). To prove (4), consider any μ^2 in

$\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*})$. Let μ^1 be a vector in $\mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})$ for which $J^{2\phi^*}\phi^{*12}\bar{\mu}^2 = \mu^1$. Then by (1)

$$J^1\phi^{21}\mu^1 = J^1\phi^{21}J^{2\phi^*}\phi^{*12}\mu^2 = J^{2\phi^*}\epsilon^2_{\phi^*}\mu^2 = \mu^2,$$

whence (4) follows. In part (5), if $\mathfrak{M}(\epsilon^1_\kappa\phi\epsilon^2) \supset \mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})$, then

$$[J^1\phi^{21}\mu^1 \mid \mu^1 \text{ in } \mathfrak{M}(\epsilon^1_\kappa\phi\epsilon^2)] \supset [J^1\phi^{21}\mu^1 \mid \mu^1 \text{ in } \mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})].$$

By (C), the left-hand side is identical with $\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*})$, which by part (2) is a subset of $\mathfrak{M}(\epsilon^2\kappa\epsilon^1_\kappa)$. The converse is obvious.

THEOREM 9. *Let ϵ^1_0 be idempotent as to ϵ^1, and κ^{12} be of type $\mathfrak{M}(\epsilon^1_0)\overline{\mathfrak{M}}(\epsilon^2)$. Suppose that ϕ^{21} is by rows-conjugate of $\mathfrak{M}(\epsilon^1_0)$ and κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^2$. Then we have the following conclusions*:

(1) *ϕ^{21} is complete by rows-conjugate of $\mathfrak{M}(\epsilon^1_0)$*;

(2) *$\mu^1 = J^2\kappa^{12}J^1\phi^{21}\mu^1$ for every μ^1 in $\mathfrak{M}(\epsilon^1_0\phi\epsilon^2)$*;

(3) $\mathfrak{M}(\epsilon^1_0\phi\epsilon^2) = [J^2\kappa^{12}\mu^2 \mid \mu^2$ *in* $\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*})] \subset \mathfrak{M}(\epsilon^1_0 \cap \epsilon^1_{\kappa^*})$;

(4) $\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*}) \subset \mathfrak{M}(\epsilon^2\kappa\epsilon^1_0)$;

(5) *if ϕ^{21} is by columns of $\mathfrak{M}(\epsilon^2)$, then $\epsilon^1_0 = J^2\kappa^{12}\phi^{21}$ and κ^{12} is complete by columns of $\mathfrak{M}(\epsilon^1_0)$*;

(6) *$\mathfrak{M}(\epsilon^1_0\phi\epsilon^2) \supset \mathfrak{M}(\epsilon^1_0 \cap \epsilon^1_{\kappa^*})$ if and only if every vector in $\mathfrak{M}(\epsilon^1_0 \cap \epsilon^1_{\kappa^*})$ is expressible in the form $J^2\kappa^{12}\mu^2$, where μ^2 is a vector in $\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*})$.*

PROOF. Part (1) follows from Theorem 7, for $\mathfrak{M}(\epsilon^2 \cap \epsilon^2_\kappa)$ is everywhere dense in $\mathfrak{M}(\epsilon^2_\kappa)$ when κ^{12} is of type $\mathfrak{M}(\epsilon^1_0)\overline{\mathfrak{M}}(\epsilon^2)$. Part (2) is proved in the same way as part (3) of Theorem 8 with the replacement of ϵ^1_κ by ϵ^1_0. By Theorem (C), part (3) is a consequence of (2). Part (4) follows from (3). For part (5), we note that ϵ^1_0 is by columns of $\mathfrak{M}(\epsilon^1_0\phi\epsilon^2)$; hence the first conclusion follows from (2) whereas the second follows from (1) and the fact that ϕ^{*12}, κ^{*21} are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^2$. Part (6) is a consequence of (3).

THEOREM 10. *Suppose that κ^{12} is of type $\mathfrak{M}(\epsilon^1)\overline{\mathfrak{M}}(\epsilon^2)$, ϕ^{21} is by rows-conjugate of $\mathfrak{M}(\epsilon^1_\kappa)$, and κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^2$. Then the following assertions are equivalent*:

(1) $\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*}) \supset \mathfrak{M}(\epsilon^2\kappa\epsilon^1_\kappa)$;

(2) *κ^{12} is complete by rows-conjugate of $\mathfrak{M}(\epsilon^2)$ and $\mathfrak{M}(\epsilon^1_\kappa\phi\epsilon^2)$ contains $\mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})$*;

(3) *κ^{12} is complete by rows-conjugate of $\mathfrak{M}(\epsilon^2)$ and $\mathfrak{M}(\epsilon^2\kappa\epsilon^1_\kappa)$ contains* $[J^1\phi^{21}\mu^1 \mid \mu^1$ *in* $\mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})]$;

(4) *κ^{*21}, ϕ^{*12} are contraceding as to $\epsilon^2\epsilon^1_{\kappa^*}\epsilon^1$, and κ^{12} is complete by rows-conjugate of $\mathfrak{M}(\epsilon^2)$*;

(5) $[J^1\phi^{21}\mu^1 \mid \mu^1 \text{ in } \mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})] \supset \mathfrak{M}(\epsilon^2\kappa\epsilon^1_\kappa)$.
If one of the preceding five conditions is valid, then $\epsilon^2 = \epsilon^2_{\kappa^*} = J^1\phi^{21}\kappa^{12}$.

Proof. The equivalence of the second and third conditions follows from (5) of Theorem 8. Conditions (1), (2), (4), and (5) are equivalent because of Theorem 2, where κ, ϕ, ϵ^1, ϵ^2, ϵ^1_ϕ are replaced by κ^*, ϕ^*, ϵ^2, ϵ^1_κ, $\epsilon^2_{\phi^*}$ respectively. The relation $\epsilon^2 = J^1\phi^{21}\kappa^{12}$ follows from (4), since κ^{12} is by columns of $\mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})$.

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