ON SPHERICAL CYCLES

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Given a metric separable space \( \mathcal{Y} \), we consider the homology group \( B^n(\mathcal{Y}) \) obtained using \( n \)-dimensional singular cycles in \( \mathcal{Y} \) with integer coefficients. Every continuous mapping \( f \in \mathcal{Y}^{S^n} \) of the oriented \( n \)-dimensional sphere \( S^n \) into \( \mathcal{Y} \) defines uniquely an element \( h(f) \) of \( B^n(\mathcal{Y}) \). Clearly if \( f_0, f_1 \in \mathcal{Y}^{S^n} \) are two homotopic mappings, then \( h(f_0) = h(f_1) \).

The homology classes \( h(f) \) will be called spherical homology classes. A cycle will be called spherical if its homology class is spherical.\(^2\)

**Theorem 1.** If \( \mathcal{Y} \) is arcwise connected, the spherical homology classes form a subgroup of \( B^n(\mathcal{Y}) \).

Let \( p \in S^n, q \in \mathcal{Y} \), and let \( S^n = S^n_+ + S^n_- \) be a decomposition of \( S^n \) into two hemispheres such that \( p \in S^n_+ \cdot S^n_- \). Consider \( f_0, f_1 \in \mathcal{Y}^{S^n} \). It is well known that, replacing if necessary \( f_0 \) and \( f_1 \) by homotopic mappings, we may assume that \( f_0(S^n_-) = q \) and that \( f_1(S^n_-) = q \). Defining \( f = f_0 \) on \( S^n_- \) and \( f = f_1 \) on \( S^n_+ \) we clearly have
\[
f \in \mathcal{Y}^{S^n}, \quad h(f) = h(f_0) + h(f_1).
\]
The homology class \( h(f_0) + h(f_1) \) is therefore spherical.

Let \( M^r \) be an \( r \)-dimensional (finite or infinite) manifold\(^3\) and \( P^{r-n-1} \) \((n > 0)\) an at most \((r-n-1)\)-dimensional subpolyhedron of \( M^r \).

**Theorem 2.** Every \( n \)-dimensional cycle \( \gamma^n \) in \( M^r - P^{r-n-1} \) such that \( \gamma^n \sim 0 \) in \( M^r \) is spherical (with respect to \( M^r - P^{r-n-1} \)).

Let \( a^{r-n-1} \) be an \((r-n-1)\)-dimensional simplex of \( M^r \) and \( b^{n+1} \) the \((n+1)\)-cell dual to it. The boundary \( \partial b^{n+1} \) is contained in \( M^r - P^{r-n-1} \) and is a spherical cycle. Since \( M^r - P^{r-n-1} \) is connected, the spherical homology classes of \( B^n(M^r - P^{r-n-1}) \) form a group. It follows that each cycle of the form
\[
(\ast) \quad \partial \left( \sum_i \alpha_i \delta_i^{n+1} \right)
\]
is a spherical cycle with respect to \( M^r - P^{r-n-1} \). The cycle \( \gamma^n \) is homologous in \( M^r - P^{r-n-1} \) to a cycle of the form \((\ast)\). Therefore \( \gamma^n \) is spherical.

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\(^1\) Presented to the Society, April 13, 1940.

\(^2\) Spherical cycles were considered by W. Hurewicz, Proceedings, Akademie van Wetenschappen te Amsterdam, vol. 38 (1935), pp. 521–528.

THEOREM 3. Let $\gamma^n$ be a spherical cycle in $M^r$ and let $r > 2n$. Then there is a simplicial homeomorphism $g \in M^{r^n}$ such that $\gamma^n \sim h(g)$.

This is an immediate consequence of Theorem 5 below. Using Theorem 2 we obtain the following:

THEOREM 4. Given an $n$-cycle $\gamma^n \subset M^r - P^{r-n-1}$ such that $\gamma^n \sim 0$ in $M^r$, there is a cycle $\gamma^n_1 \subset M^r - P^{r-n-1}$ which is a simplicial and homeomorphic image of $S^n$ such that $\gamma^n \sim \gamma^n_1$ in $M^r - P^{r-n-1}$.

THEOREM 5. Let $Q^n$ be a finite $n$-dimensional polyhedron and let $r > 2n$. Every continuous mapping $f \in M^{Q^n}$ can be approached by simplicial homeomorphisms $g \in M^{Q^n}$.

We may admit that the mapping $f$ is simplicial. Let $a_1, a_2, \cdots, a_k$ be the vertices of the complex $f(Q^n)$ and let $\sigma_1, \sigma_2, \cdots, \sigma_k$ be the corresponding stars. Let us choose $\delta > 0$ so that $x \in f(Q^n)$ will imply $\rho(x, M^r - \sigma_i) > \delta$ for some $i = 1, 2, \cdots, k$.

Let $\delta > 2\epsilon > 0$. We are going to define a sequence $f = f_0, f_1, \cdots, f_k$ of simplicial maps of $Q^n$ into $M^r$ such that

1. $\left| f_i(x) - f_{i-1}(x) \right| < \frac{\epsilon}{k}$,
2. $f_i(x_1) = f_i(x_2)$ implies $f_{i-1}(x_1) = f_{i-1}(x_2)$,
3. $x_1 \neq x_2$ and $f_i(x_1) = f_i(x_2) = y$ imply $\rho(y, M^r - \sigma_i) < \delta \frac{2k - i}{2k}$.

Suppose that $f_0, f_1, \cdots, f_{i-1}$ are already defined. Let

$$f_i(x) = f_{i-1}(x) \quad \text{if} \quad f_{i-1}(x) \in M^n - \sigma_i,$$
and let $Q^n_i = f_{i-1}^{-1}(\sigma_i)$.

$M^r$ being a manifold, $\sigma_i$ is simplicially homeomorphic with a convex $r$-cell in a euclidean $r$-dimensional space. Since $r > 2n$, then using the very well known procedure of making vertices linearly independent we find a simplicial map $f_i(Q^n_i) \subset \sigma_i$ such that $f_i(x) = f_{i-1}(x)$ if $f_{i-1}(x)$ is on the boundary of $\sigma_i$ and satisfying (1)–(3).

Taking $g = f_k$ it follows from (1) that

$$\left| g(x) - f(x) \right| < \epsilon.$$

4 With respect to certain simplicial subdivisions of $M^r$ and $S^n$.
5 $\sigma_i$ consists of all closed simplices of $M^r$ containing $a_i$.
Now if $x_1 \neq x_2$ and $g(x_1) = g(x_2)$, then according to (2) we have

$$f_i(x_1) = f_i(x_2) = y_i \quad \text{for } i = 0, 1, \cdots, k.$$ 

Owing to the definition of $\delta$ there is an index $j = 0, 1, \cdots, k$ such that

$$\rho(y_0, M^n - \sigma_j) > \delta.$$ 

Combining this with (1) we see that

$$\rho(y_i, M^n - \sigma_j) > \delta - \frac{\epsilon i}{k} > \delta - \frac{\delta i}{2k} = \delta - \frac{2k - i}{2k}.$$ 

Taking $i = j$ we obtain a contradiction with (3).