

THE DOUBLE COSETS OF A FINITE GROUP¹

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1. **Introduction.** It is the purpose of this paper to study some of the properties of the double cosets of a finite group and to prove two main theorems which generalize the results of two previous papers by the author,² giving some relations between the double cosets and the irreducible components of the permutation group generated by a given subgroup. We let H be an arbitrary but fixed subgroup of order h of a finite group G of order g , $g = nh$, and we let G_H be the permutation group of degree n induced by right multiplication of the cosets HS_i , $i = 1, 2, \dots, n$, by elements of G . When written as a group of permutation matrices and completely reduced, the group G_H will have r' distinct irreducible components Γ_i of degree n_i and multiplicity μ_i^H , and we may write

$$(1.1) \quad G_H = \sum_{i=1}^{r'} \mu_i^H \Gamma_i.$$

Multiplication of a right coset HG_k on the left by a single element of G does not in general produce a right coset, but if each coset HG_k is multiplied on the left by all the elements of a right coset HS_t and the products are added, a transformation is obtained which carries each of the n right cosets HG_k into a collection of right cosets $\sum_{\theta=1}^h HS_t H_\theta G_k$ in which, as we shall see in §4, each of the k_t cosets occurs $d_t = h/k_t$ times. Its matrix $d_t V_t$ is permutable with each of the matrices of G_H . Certain cosets, which we shall call associated cosets, are permuted among themselves when multiplied on the right by elements of H . Each of these produces the same matrix V_t . The totality K_t of elements belonging to a complete set of k_t associated cosets, each counted once, will be called a *double coset*, whereas the term *weighted double coset* will refer to the complex of h^2 elements $HS_t H$ in which each element of the double coset K_t occurs $d_t = n/k_t$ times. The integer d_t will be called the density. The number of distinct double cosets K_t will be denoted by r , and the elements S_t , $t = 1, 2, \dots, r$, one from each, will be said to *generate* the double

¹ Presented to the Society, September 12, 1940.

² J. S. Frame, *The degrees of the irreducible components of simply transitive permutation groups*, Duke Mathematical Journal, vol. 3 (1937), pp. 8–17.

J. S. Frame, *On the decomposition of transitive permutation groups generated by the symmetric group*, Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 132–139.

cosets. The double coset which contains the inverses of the elements of K_t will be denoted by $K_{t'}$.

In §2 a number of elementary properties of double cosets are stated, mostly without proof. In §3 a study is made of self-inverse double cosets, $K_t = K_{t'}$, and their number is shown to equal $(1/g)\sum\chi(R^2)$, where $\chi(R^2)$ is the trace of R^2 in the permutation group G_H , and where the sum is taken for all R in G . Using the results of a paper of Frobenius³ we then prove the first principal theorem, which generalizes a result obtained by the author for the symmetric group.⁴

THEOREM A. *The number of self-inverse double cosets of a finite group G with respect to a subgroup H is equal to the sum of the multiplicities of those irreducible components of G_H which have a symmetric bilinear invariant minus the sum of the multiplicities of those which have an alternating bilinear invariant.*

The Hermitian invariants of G_H associated with the matrices V_t are studied in §4.⁵ Two bases consisting of r independent invariants are found, the one obtained directly from the r double cosets, and the other from the $r = \sum_{t=1}^r (\mu_t^H)^2$ Hermitian invariants⁶ which come into evidence when G_H is completely reduced. The complex multiplication of the double cosets plays an important role in the discussion which culminates in the proof of Theorem B, which includes as a special case a theorem conjectured but only partially proved in a previous paper.⁷

THEOREM B. *Given a transitive permutation group G_H of degree n in which the subgroup H leaving one symbol fixed permutes the n symbols in r transitive sets of k_t symbols, $t = 1, 2, \dots, r$. Let K be the product $\prod_{t=1}^r (k_t)$ and let N , $N = \prod_{t=1}^r n_t^{(\mu_t^H)^2}$, denote the product of the degrees n_t of the distinct irreducible components Γ_t of G_H each raised to a power equal to the square of its multiplicity in G_H . Then $n^{r-2}K/N = \bar{P}_1 P_1$, where P_1 is an algebraic integer in the field of the characters of the components of G_H .*

³ G. Frobenius and I. Schur, *Über die reellen Darstellungen der endlichen Gruppen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1906 (I), pp. 186–208, especially p. 197. The author is indebted to G. de B. Robinson for referring him to this article.

⁴ J. S. Frame, Proceedings of the National Academy of Sciences, loc. cit.

⁵ A study of these matrices was made by I. Schur, *Zur Theorie der einfach transitiven Permutationsgruppen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1933, pp. 598–623.

⁶ This relation is well known in group theory. See, for example, W. Burnside, *Theory of Groups*, 1911, p. 275.

⁷ J. S. Frame, *Duke Mathematical Journal*, loc. cit.

In §5 a study of the bilinear invariants of the components of G_H which are irreducible in the real domain leads to an alternate proof of Theorem A, based on methods similar to those of §4.

2. Elementary properties of double cosets. The set of elements permutable with a coset HS_i form a group N_i called the normalizer of the coset, whose intersection with H is a group D_i of order d_i leaving HS_i fixed in the permutation group G_H . We let D_iH_{ij} be the $k_i = h/d_i$ cosets of H with respect to D_i and note that for each j the elements $HS_iD_iH_{ij} = H(S_iH_{ij})$ form a different coset HS_j associated with HS_i . Hence there are k_i distinct cosets each occurring d_i times in the weighted double coset HS_iH . If H is invariant in G , then each $D_i = H$, each $k_i = 1$, and each coset is a double coset. It is convenient to select the S_j for associated cosets so that $S_iH_{ij} = S_j$ and then to define $H_{ji} = S_j^{-1}S_i = H_{ij}^{-1}$, when obtaining the cosets of H with respect to its subgroup $D_j = H_{ij}^{-1}D_iH_{ij}$, which is permutable with HS_j . It is seen immediately that the subgroups D_i, D_j, \dots which are permutable with the right cosets HS_i, HS_j, \dots of a double coset HS_iH are all of the same order d_i and form a complete set of conjugate subgroups of H .

The following four properties of double cosets are simple enough to be given here without proof.⁸

THEOREM 2.1. *Each element of G lies in one and only one double coset of G with respect to a given subgroup H .*

THEOREM 2.2. *A double coset may be generated by any one of its elements.*

THEOREM 2.3. *The inverses $H_i^{-1}S_i^{-1}H_i^{-1}$ of the elements $H_iS_iH_i$ of the double coset K_i form a double coset $K_{i'}$ generated by S_i^{-1} , which has the same density as K_i .*

If $K_i = K_{i'}$, the double coset is called self-inverse.

THEOREM 2.4. *A double coset which contains a self-inverse element is self-inverse. In particular the double coset $H = K_1$ is self-inverse.*

The next three theorems show that the elements of a class of conjugates, of a left coset, and of the set of inverses of a right coset, are equally distributed among the right cosets of their double coset.

THEOREM 2.5. *Each coset of a double coset K_i contains the same num-*

⁸ Several of the theorems in §2 are implied in the discussion of cosets in the standard texts on group theory. Nowhere has the author found them collected as properties of double cosets as such. They are stated here for convenience of reference for the later proofs.

ber $g_{i\lambda}$ of the g_λ elements forming the class C_λ of conjugate elements in G .⁹

PROOF. Let $H_\theta S_i$ be a common element of HS_i and C_λ , and let the subgroup of D_i which is permutable with the element $H_\theta S_i$ be of index $g_{i\lambda}$ in D_i . Then there will be $g_{i\lambda}$ distinct elements each counted $d_i/g_{i\lambda}$ times among the elements $\delta_\kappa^{-1}(H_\theta S_i)\delta_\kappa$, δ_κ in D_i . Each of these distinct elements of HS_i is transformed by H_{i_j} into a different element of HS_j , so the numbers $g_{i\lambda}$, $g_{j\lambda}$, \dots are the same for all the cosets of K_i and may be denoted by $g_{i\lambda}$.

THEOREM 2.6. *The h elements of the left coset $(H_\theta S_i)H$ are equally distributed among the k_i right cosets associated with HS_i , just d_i elements lying in each right coset.*

PROOF. Each element of H can be written uniquely in the form $\delta_\kappa H_{i_j}$, where δ_κ is in D_i . For each j we obtain d_i elements $H_\theta S_i \delta_\kappa H_{i_j} = (H_\theta S_i \delta_\kappa S_i^{-1}) S_j$ in HS_j .

If we note further that the inverses of the elements of a right coset HS_i form a left coset $S_i^{-1}H$, then we obtain the following:

THEOREM 2.7. *The inverses of the elements of a right coset are equally distributed among the k_i cosets of the inverse double coset, with d_i elements in each. Furthermore each coset of a self-inverse double coset K_i contains just d_i of its own inverses.*

THEOREM 2.8. *Each element of the normalizer N_1 of H transforms a double coset K_i into itself or into a double coset $N_1^{-1}K_i N_1$ having the same distribution of elements among the classes C_λ as K_i does.*

These double cosets may be called conjugate double cosets.

PROOF. Each element is transformed into a conjugate element in the double coset generated by $N_1^{-1}S_i N_1$.

THEOREM 2.9. (a) *The number of times, k_{tuv} , that the identity element E occurs among the h^3 elements $HS_t HS_u HS_v$ depends only on the double cosets to which S_t , S_u , and S_v belong.*

(b) *The constant k_{tuv} is unchanged by an even permutation of the double cosets K_t , K_u , K_v , or by an odd permutation coupled with a change to inverse double cosets.*

(c) *The number of the h elements $S_u HS_v$ which lie in the double coset $K_{t'}$ is k_{tuv}/d_t , and $\sum_{t=1}^r k_{tuv}/d_t = h$.*

(d) *For $t=1$, we have $k_{1uv} = k_{uv1} = hd_u \delta_{uv}$.*

⁹ D. E. Littlewood, *Theory of Group Characters*, 1940, p. 149. Littlewood shows, if we change his notation to ours, that the characters in G_H of the elements of C_λ are $ng_{i\lambda}/g_\lambda$.

(e) *The product of the double cosets K_tK_u is given by the equation*

$$(2.10) \quad K_tK_u = (h/(d_t d_u)) \sum_v k_{tv} K_v.$$

To prove the various parts of Theorem 2.9, let us assume that $H_a, H_b,$ and H_c are elements of H which satisfy the equation $H_a S_t H_b S_u H_c S_v = E$. The equation is still valid if we permute the six elements cyclicly, or if we interpose between two adjacent ones the product of an element of H and its inverse, thus replacing an element by another element in its double coset, or if we replace all six elements by their inverses in the opposite order. The remainder of the proof depends on eliminating the weighting factors from the weighted double cosets $HS_tH,$ etc., but may be left to the reader.

3. The number of self-inverse double cosets. We shall now apply the theorems of §2 to obtain expressions for the number of self-inverse double cosets of a group.

THEOREM 3.1. *The number N_H of self-inverse double cosets K_t of G with respect to H is $1/g$ times the number of solutions of the equation*

$$S_j R^2 = H_k S_j, \quad j = 1, 2, \dots, n; H_k \text{ in } H, R \text{ in } G.$$

We rewrite the equation in the form $H_k^{-1}(S_j R S_j^{-1}) = (S_j R S_j^{-1})^{-1}$, which states that the inverse of the element $S_j R S_j^{-1} = T$ belongs to its own right coset. For each of the k_t cosets of K_t there are d_t available values of T , by Theorem 2.7. For each of these $h = d_t k_t$ values of T we may choose n generators S_j to determine an element $R = S_j^{-1} T S_j$ conjugate to T . Thus for each self-inverse double coset we obtain $g = nh$ solutions of the given equation, and the theorem is proved.

Now the permutation matrix corresponding to R^2 in G_H has a unit in the principal diagonal for every right coset which remains fixed by R^2 , that is, for every solution of the equation $S_j R^2 = H_k S_j, H_k \text{ in } H, j = 1, 2, \dots, n$. Its trace is the number of these solutions for fixed R . When we sum these for all R in G and apply Theorem 3.1, we obtain the following theorem.

THEOREM 3.2. *The number N_H of self-inverse double cosets of G with respect to H is given by the formula*

$$(3.3) \quad N_H = (1/g) \sum_{R \text{ in } G} \chi(R^2),$$

where $\chi(S)$ is the trace of the matrix S in the permutation group G_H .

This theorem may be applied to the results of a paper by Frobenius

and Schur¹⁰ to obtain the fundamental Theorem A of §1. It is known in the theory of group characters that for an irreducible representation Γ_i with character $\chi^i(R)$ the quantity

$$(3.4) \quad \sum_R \{ [\chi^i(R)]^2 + \chi^i(R^2) \} / (2g)$$

is 1 or 0 according as Γ_i has or has not a symmetric bilinear invariant, and that

$$(3.5) \quad \sum_R \{ [\chi^i(R)]^2 - \chi^i(R^2) \} / (2g)$$

is 1 or 0 according as Γ_i has or has not an alternating bilinear invariant. Following Frobenius¹⁰ we set $c_i = 1, -1, \text{ or } 0$ according as Γ_i has a symmetric, an alternating, or no bilinear invariant. We shall call these respectively "symmetric," "quaternion," and "rotary" representations. Then

$$(3.6) \quad (1/g) \sum_R \chi^i(R^2) = c_i.$$

Now let G_H be reduced into irreducible components Γ_i with multiplicities μ_i^H . Then

$$(3.7) \quad G_H = \sum_i \mu_i^H \Gamma_i, \quad \chi(R) = \sum_i \mu_i^H \chi^i(R).$$

Hence, by (3.3), (3.7), and (3.6), we have

$$(3.8) \quad N_H = (1/g) \sum_R \chi(R^2) = (1/g) \sum_R \sum_i \mu_i^H \chi^i(R^2) = \sum_i c_i \mu_i^H.$$

This formula is equivalent to Theorem A of §1.

4. A theorem derived from a unitary reduction of a group and its double cosets. When a group G is represented in two ways as a regular permutation group of degree g , using cosets with respect to the identity subgroup E , the one G_E obtained by the right multiplication of cosets by the elements of G , and the other G'_E by left multiplication, the matrices of G'_E form a basis for all matrices permutable with those of G_E , and vice versa. In that case each of the elements of G forms a double coset. But when cosets are taken with respect to a subgroup $H \neq E$, a basis for the matrices permutable with those of the right multiplication permutation group G_H is found in the double cosets of the left multiplication group. The matrices V_i described in §1 are

¹⁰ G. Frobenius and I. Schur, loc. cit.

those of a complete set of invariant Hermitian forms of G_H on which all others are linearly dependent. For let $x_i, x_i = x(HS_i), i = 1, 2, \dots, n$, be the variables permuted by G_H . Then the matrix R of G_H transforms the product $\bar{x}(H)x(HS_i)$ into $\bar{x}(HR)x(HS_iR)$, and if we write $R = H_jG_k$ and sum over all R in G , we obtain an Hermitian form invariant under the right multiplications of G_H . Since the subgroup D_i of H leaves HS_i fixed, each term of this form will contain the factor d_i . Dividing by d_i and writing $K_i = HS_iH/d_i$, we obtain the simpler invariant Hermitian form $\sum_k \bar{x}(HG_k)x(K_iG_k)$, whose matrix we have denoted by V_i . The matrix V_1 is the unit matrix. Each row and column of V_i contains k_i 1's and the rest 0's, and the sum of all the matrices V_i is a matrix consisting entirely of 1's.¹¹ To the inverse double coset K_i' corresponds the transposed matrix V_i' , which will be denoted by V_i' . Self-inverse double cosets have symmetric matrices V_i .

Now let U be a unitary matrix which completely reduces the group G_H into its irreducible components Γ_i of degree n_i and multiplicity μ_i^H , so that all equivalent components of $U^{-1}G_HU$ are actually identical and so that the invariant Hermitian form for each component is a diagonal form with unit matrix E_i of degree n_i . Then for the set of μ_i^H equal components Γ_i we have $(\mu_i^H)^2$ linearly independent Hermitian forms whose matrices $\sum_{\alpha\beta} E_i^{\alpha\beta} \rho_i^{\alpha\beta}$ are obtained as the direct product of E_i with an arbitrary matrix $(\rho_i^{\alpha\beta})$ of degree μ_i^H . The r matrices $M_i = U^{-1}V_iU$, obtained by transforming the Hermitian invariants of G_H , must be expressible as linear combinations of the r matrices $E_i^{\alpha\beta}$. It is convenient to arrange the symbols $E_i^{\alpha\beta}$ in some arbitrary order starting with $E_i^{11} = E_i$, and to assign to each a single subscript γ . We write

$$(4.1) \quad M_i = \sum_{\gamma} E_{\gamma} \rho_{\gamma i}; \quad \text{where } E_{\gamma} = E_i^{\alpha\beta} \quad \text{and } \rho_{\gamma i} = \rho_i^{\alpha\beta}(K_i);$$

$$\gamma, i = 1, 2, \dots, r.$$

Since the matrices M_i and E_{γ} each form a basis for the invariant Hermitian forms of $U^{-1}G_HU$, the matrix $(\rho_{\gamma i})$ is nonsingular.

The M_i combine according to a rule similar to the rule of combination for the inverse double cosets K_i' , as given in 2.10, namely,

$$(4.2) \quad M_i M_u = \sum_v c_{iuv} M_v, \quad c_{iuv} = k_{i' u' v'} / (d_i d_u).$$

Since this is also the rule of combination for the matrices $\rho_i^{\alpha\beta}(K_i)$ corresponding to a given Γ_i , these matrices give that representation for the left multiplication of double cosets which is associated with Γ_i .

¹¹ J. S. Frame, Duke Mathematical Journal, loc. cit.

Now when G_E and $G_{E'}$ are simultaneously reduced, the component of $G_{E'}$ on the same variables as the n_i components Γ_i of G_E consists of the direct product of E_i with a representation equivalent to Γ'_i . When the matrices of this representation are summed over all elements in each double coset, the resulting set of r matrices when reduced will contain the component of degree $\mu_i^H \leq n_i$, expressible in terms of the variables of G_H , which is equivalent to the matrices $\rho_i^{\alpha\beta}(K_i)$. Hence the $\rho_i^{\alpha\beta}(K_i)$ are expressible in terms of the matrices of Γ'_i .

The rule (4.2) is also the rule of combination for the matrices V_t ,¹² so the coefficients c_{tuv} must be integers, and the character of the product $M_{t'}M_u$ is given by the formula

$$(4.3) \quad \chi(M_{t'}M_u) = nc_{t'u1} = nk_t\delta_{tu},$$

in view of Theorem 2.9 (d). Since U is unitary, the matrix $M_{t'}$ is the transposed conjugate \bar{M}'_t of M_t . We shall denote by $E_{\gamma'}$ the transposed conjugate of E_γ , and by n_γ the degree of the corresponding irreducible representation.

Next consider the matrices $(M_{t'u})$ and $(E_{\gamma'\delta})$ of degree r whose elements are

$$(4.4) \quad M_{t'u} = \chi(M_{t'}M_u) = \chi(V_{t'}V_u) = nk_t\delta_{tu},$$

$$(4.5) \quad E_{\gamma'\delta} = \chi(E_{\gamma'}E_\delta) = n_\gamma\delta_{\gamma\delta}.$$

We obtain a relation between $(M_{t'u})$ and $(E_{\gamma'\delta})$ as follows:

$$(4.6) \quad M_{t'u} = \chi\left(\sum_\gamma \bar{p}_{t\gamma}E_{\gamma'} \sum_\delta E_\delta\rho_{\delta u}\right) = \sum_{\gamma,\delta} \bar{p}_{t\gamma}E_{\gamma'\delta}\rho_{\delta u}.$$

Denoting the determinant of $(\rho_{\delta u})$ by P , we have, by (4.4), (4.5), (4.6),

$$(4.7) \quad \prod_{t=1}^r (nk_t) = \left(\prod_{\gamma=1}^r n_\gamma\right) \bar{P}P = \left(\prod_{i=1}^{r'} n_i^{(\mu_i^H)^2}\right) \bar{P}P.$$

Since $\sum_t \rho_{\gamma t} = n\delta_{\gamma 1}$, the determinant P may be written in the form nP_1 , where P_1 is the minor of ρ_{11} in P . Factoring n^2 from both sides of (4.7) and using the notation of Theorem B, we have

$$(4.8) \quad n^{r-2}K = N\bar{P}_1P_1, \quad \text{where} \quad K = \prod_{t=1}^r k_t, \quad N = \prod_{i=1}^{r'} n_i^{(\mu_i^H)^2}.$$

Since for each i the matrices $\rho_i^\alpha(K_i)$ give a representation of the ring of matrices V_i in which the coefficients of combination are integers, the matrix P_1 is an algebraic integer. It belongs to the field of char-

¹² J. S. Frame, Duke Mathematical Journal, loc. cit.

acters of the Γ_i . This completes the proof of Theorem B. It will be noted that if the representations Γ_i have rational characters, which will certainly be true if their degrees are all distinct, then the quotient $n^{r-2}K/N$ will be a perfect square.

5. A theorem derived from a real orthogonal reduction of a group and its double cosets. Let us now consider the reduction of G_H by a real orthogonal matrix O into a form which is irreducible in the real domain. The component representations are of three types: (1) Those “symmetric” representations $\Gamma_i^{(+)}$ which are absolutely irreducible in the complex domain and have a symmetric bilinear invariant. (2) Those “quaternion” representations $2\Gamma_i^{(-)}$ which consist of two equivalent complex components of even degree each with real characters and each with an alternating but not a symmetric bilinear invariant. The pair together have a third alternating bilinear invariant and a symmetric bilinear invariant. The matrices of the four invariants, suitably normalized, combine like the quaternion units. (3) Those “rotary” representations $\Gamma_i^{(0)} + \Gamma_i^{(0)}$ which have two non-equivalent conjugate complex absolutely irreducible components, each having complex characters, but having no bilinear invariant. Taken together they have an alternating and a symmetric bilinear invariant which, when suitably normalized, combine like the real and imaginary units.

The matrices $O^{-1}V_iO$ form a basis for the invariant bilinear forms, but in place of the other basis matrices E_γ used in §4, we now use matrices E_γ which for the symmetric representations of type 1 are like the old E_γ , for the quaternion representations of type 2 come in sets of four which multiply like the quaternion units, and for the rotary representations of type 3 come in pairs which multiply like the real and imaginary units. Those of type 2 are as follows:

$$(5.1) \quad \begin{pmatrix} e_i & 0 & 0 & 0 \\ 0 & e_i & 0 & 0 \\ 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & e_i \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & e_i \\ -e_i & 0 & 0 & 0 \\ 0 & -e_i & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -e_i & 0 & 0 \\ e_i & 0 & 0 & 0 \\ 0 & 0 & 0 & e_i \\ 0 & 0 & -e_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & e_i \\ 0 & 0 & -e_i & 0 \\ 0 & e_i & 0 & 0 \\ -e_i & 0 & 0 & 0 \end{pmatrix},$$

where e_i denotes the unit matrix of degree $n_i/2$.

We next normalize each of the matrices V_t and E_γ by dividing by the square root of the sum of the squares of its coefficients. From the new normalized matrices V_t^* and E_γ^* , thought of as of degree n , we form a new set of matrices of degree r defined by

$$(5.2) \quad \begin{aligned} V_{tu}^* &= \chi(V_t^* V_u^*), & V_{t'u}^* &= \chi(V_t^{*'} V_u^*), \\ E_{\gamma\delta}^* &= \chi(E_\gamma^* E_\delta^*), & E_{\gamma'\delta}^* &= \chi(E_{\gamma'}^* E_\delta^*). \end{aligned}$$

Then (V_{tu}^*) and $(E_{\gamma\delta}^*)$ are each the unit matrix, whereas $(V_{t'u}^*) = (\delta_{t'u})$ is a matrix whose trace is the number N_H of self-inverse double cosets, and $(E_{\gamma\delta}^*)$ is a matrix whose trace is $\sum_{i=1}^r c_{i\delta} \mu_i^H$, since each component $\Gamma_i^{(+)}$ of type 1 contributes $+1$, each real component $2\Gamma_i^{(-)}$ of type 2 containing a pair of equivalent irreducible representations contributes $+1 - 1 - 1 - 1 = -2$, and each real component of type 3 contributes $+1 - 1 = 0$. Now let

$$(5.3) \quad O^{-1} V_t^* O = \sum_{\gamma} E_{\gamma}^* q_{\gamma t}$$

define a matrix Q which changes the basis of the bilinear forms of $O^{-1} G_H O$. Then

$$(5.4) \quad \begin{aligned} V_{tu}^* &= \chi(V_t^* V_u^*) = \chi(O^{-1} V_t^* O \cdot O^{-1} V_u^* O) = \chi \left(\sum_{\gamma} E_{\gamma}^* q_{\gamma t} \cdot \sum_{\delta} E_{\delta}^* q_{\delta u} \right) \\ &= \sum_{\gamma, \delta} q_{\gamma t} E_{\gamma\delta}^* q_{\delta u}. \end{aligned}$$

Similarly, since $(V_{t'u}^*)$ and $(E_{\gamma\delta}^*)$ are unit matrices, we have

$$(5.5) \quad V_{t'u}^* = \sum_{\gamma, \delta} q_{\gamma' t'} E_{\gamma\delta}^* q_{\delta u} = \sum q_{\delta t'} q_{\delta u} = \delta_{t'u}.$$

Hence Q is an orthogonal matrix, and (5.4) may be written in the form

$$(5.6) \quad (V_{tu}^*) = Q^{-1} (E_{\gamma\delta}^*) Q.$$

By equating the traces of (V_{tu}^*) and $(E_{\gamma\delta}^*)$ given above, an alternate proof of Theorem A is obtained.

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