CONDITIONS FOR THE CONTINUITY OF ARC-PRESERVING TRANSFORMATIONS[1]

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1. **Introduction.** A single-valued transformation $T(A)=B$, where A and B are topological spaces, is said to be *arc-preserving*[2] provided that the image of every simple arc in A is either a simple arc or a a single point in B. Even when A is a simple arc, an arc-preserving transformation may fail to be continuous; for example: on the unit interval $(x_0=0\leqq x\leqq 1=x_1)$ let $x_n=1/n$ $(n=1, 2, 3, \cdots)$. Define $T(x_0)=x_0$ and for each interval A_n $(x_{n+1}\leqq x\leqq x_n)$ let $T(A_n)=A$ be a topological transformation such that $T(x_n)=x_0$ or x_1 according as n is even or odd. Then the transformation $T(A)=A$ is arc-preserving, but fails to be continuous at x_0.

The results of this paper concern conditions under which an arc-preserving transformation is continuous, and the conclusions lead to homeomorphisms. We consider only the case where A is a locally connected continuum. The transformation T may be made continuous by putting conditions on the space A or by putting added conditions on the transformation T itself. In this paper we take both points of view. We shall say that A is *strongly arcwise connected* provided every infinite subset of A intersects some arc of A in infinitely many points. Our principal theorem states that if A is cyclic and $T(A)=B$ is arc-preserving then T will be topological or B will be an arc provided either A is strongly arcwise connected or T is tree-preserving[3] (that is, the image of every tree in A is a tree or a single point in B). Moreover, we show that if B is not an arc then A must be strongly arcwise connected in order that a topological mapping be the only arc-preserving transformation of A onto B.

Throughout the paper A is a locally connected continuum and T is a single-valued transformation, but not necessarily continuous. It is understood that a single point is to be regarded as an arc.

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[1] Presented to the Society in parts as follows: April 6, 1940, under the title *On arc-preserving transformations*, by Puckett; April 26, 1940, under the title *On arc and tree preserving transformations*, by D. W. Hall; and September 12, 1940, under the title *Arc-preserving transformations of a certain class of spaces*, by Hall and Puckett.

[2] See G. T. Whyburn, *Arc-preserving transformations*, American Journal of Mathematics, vol. 58 (1936), pp. 305–312. See also D. W. Hall and G. T. Whyburn, *Arc- and tree-preserving transformations*, Transactions of this Society, vol. 48 (1940), pp. 63–71.

[3] See R. G. Simond, Duke Mathematical Journal, vol. 4 (1938), pp. 575–589; also Hall and Whyburn, loc. cit.

2. **On strongly arcwise connected sets.** The set A is said to be *strongly arcwise connected* provided that every infinite subset of A contains infinitely many points which lie on an arc in A. The following are immediate consequences of this definition:

(2.1) *A strongly arcwise connected set is compact and locally connected.*

(2.2) *In order that a continuum be strongly arcwise connected it must be the sum of a finite number of cyclic chains.*

(2.3) *The property of a continuum's being strongly arcwise connected is cyclicly reducible.*[4]

(2.4) *The property of being strongly arcwise connected is invariant under an arc-preserving transformation.*

(2.5) *If A is strongly arcwise connected and $T(A)=B$ is a one-to-one arc-preserving transformation, then T is topological.*

Proof. We need only show that $T(A)=B$ is continuous. To this end let $\{x_n\}$ be a sequence of points converging to a point x in A. Since A is strongly arcwise connected we lose no generality in assuming that all the points $\{x_n\}$ lie on an arc α in A. For each n let α_n be the irreducible subarc of α containing $x+\sum_{i=n}^{\infty}x_i$; then $\prod\alpha_n=x$. Now $T(\prod\alpha_n)=\prod T(\alpha_n)$, since T is one-to-one. Thus $T(x)=\prod T(\alpha_n)$ is the intersection of a monotone decreasing sequence of arcs. Thus $T(x_n)$ converges to $T(x)$, since $T(x_n)$ is contained in $T(\alpha_n)$ for every n.

The following example shows that condition of strong arcwise connectivity cannot be omitted:

Example. *There exists a one-to-one arc-preserving transformation $T(A)=B$, where A is a cyclicly connected continuum, which is not continuous.*

Proof. The example will be constructed in the euclidean plane. Let L be the unit interval and for every positive integer n let A_n be a line segment of length $1/n^2$ erected perpendicular to L at the point $1/n$. Define a_n as the end of A_n not on the line L and let B_n be the segment joining the point a_n to the origin. Then the cyclicly connected continuum of the example will consist of the unit interval L together with all the segments A_n and B_n.

To construct B let O denote the origin and for every positive integer n define Q_n as the point $(1/n, 0)$. Let M denote the line $x=.7$, and $P_1=(1, 1)$. We may then define an infinite sequence of points

[4] See Kuratowski and Whyburn, Fundamenta Mathematicae, vol. 16 (1930), pp. 305–331.

$\{P_n\}$ as follows: assuming that P_{n-1} has been defined let R_{n-1} be the point of intersection of the bisector of the angle $OP_{n-1}Q_{n-1}$ with the line M, and define P_n as the midpoint of the segment $P_{n-1}R_{n-1}$. Let C_n be the segment OP_n and D_n the segment P_nQ_n. The set B is then defined as the sum of L and all the line segments C_n and D_n.

The transformation $T(A)=B$ is now easily set up. Define $T(x)=x$ for all x in L, and let T send the sum of A_n and B_n topologically into the sum of C_n and D_n. Evidently $T(A)=B$ as thus defined is one-to-one and arc-preserving, but not continuous at the origin.

(2.6) *Any locally connected continuum A which is not strongly arcwise connected may be mapped onto the unit circle by an arc-preserving transformation.*

Proof. Since A is not strongly arcwise connected it contains a sequence of disjoint regions[5] $\{U_n\}$ such that no arc in A intersects infinitely many of the U_n. Clearly, the set $M=A-\sum U_n$ and the points of $\sum U_n$ give an upper semi-continuous decomposition of A. This decomposition determines a continuous transformation $T_1(A)=A_1$ such that each $T_1(U_n)$ is a component of $A_1-T_1(M)=A_1-p_1$ and $A_1=p_1+\sum T_1(U_n)$. Now for each positive integer n let L_n be the line segment in the euclidean plane between the points $x_0=(0, 0)$ and $x_n=(1/n, 1/n^2)$ and define $A_2=\sum L_n$. Then there exists a continuous transformation $T_2(A_1)=A_2$ such that $T_2(\overline{T_1(U_n)})=T_2(T_1(U_n)+p_1)=L_n$. Finally, let B be the unit circle $x=\cos\theta$, $y=\sin\theta$ $(0\leqq\theta<2\pi)$ and let B_n be the subarc of B given by $0\leqq\theta\leqq(2n-1)\pi/n$. A sequence of topological transformations $T_3(L_n)=B_n$ which in every case maps x_0 onto the point $(1, 0)$ of B defines a transformation $T_3(A_2)=B$. Let α be any arc of A and consider its image $T(\alpha)=T_3T_2T_1(\alpha)$ in B. Since $T_2T_1(A)=A_2$ is continuous and α can intersect at most a finite number of the regions U_n, it follows that $T_2T_1(\alpha)$ is a connected subset of A_2 and is contained in some $\sum_{i=1}^{k}L_{n_i}$ $(n_{i+1}>n_i)$. Moreover, by construction $T_3(\sum_{i=1}^{k}L_{n_i})=B_{n_k}$ is continuous and, consequently, $T_3T_2T_1(\alpha)$ is an arc. Thus $T(A)=T_3T_2T_1(A)=B$ is arc-preserving. However, it will be noted that T fails to be continuous at any point in the limit superior of $\{U_n\}$, but is continuous on every arc α of A.

3. **Lemmas.** In this section we obtain some preliminary results concerning arc-preserving transformations defined on cyclic locally connected continua. The set B is a topological space.

(3.1) *Let A be cyclic, let $T(A)=B$ be arc-preserving, and let $G=[\alpha]$*

[5] See Hall and Puckett, *Strongly arcwise connected spaces*, to appear in American Journal of Mathematics.

be the set of all arcs with endpoints in $T^{-1}(p')$ and $T^{-1}(q')$ respectively. If the image of every simple closed curve of A is an arc, then $\prod T(\alpha)$ contains an arc joining p' and q'.

PROOF. Let $G^*=[\alpha^*]$ be the subcollection of arcs of G which have $p^*\in T^{-1}(p')$ and $q^*\in T^{-1}(q')$ for their endpoints, and let α_1^* and α_2^* be any two arcs of this collection. Now let β_1' and β_2' be subarcs of $T(\alpha_1^*)$ and $T(\alpha_2^*)$ respectively which have p' and q' as endpoints. Suppose there exists a point x' of β_1' not contained in β_2'; then $\beta_1'+\beta_2'$ contains a simple closed curve J'. Therefore, since $T^{-1}(x')$ is disjoint with α_2^*, $\alpha_1^*+\alpha_2^*$ contains a simple closed curve J such that $T(J)$ contains J', contrary to hypothesis. Define $\beta'=\beta_1'=\beta_2'$, and suppose there exists a point $p\neq p^*$ of $T^{-1}(p')$. Since A is cyclic it contains an arc $p^*q^*+q^*p$. Now $T(p^*q^*)$ contains β', since p^*q^* is an arc of G^*. Moreover, $T(q^*p)$ must contain β', for otherwise $T(p^*q^*+q^*p)$ would contain a simple closed curve. Because of the symmetry of the above argument it follows that $\prod T(\alpha)$ contains β'.

(3.2) *If $T(A)=B$ is arc-preserving and J is a simple closed curve, then $T(J)=J'$ is topological or J' is an arc.*

PROOF. Suppose J' is not an arc. Then, since J is strongly arcwise connected, we need only show that $T(J)=J'$ is one-to-one, by virtue of (2.5). If $T(J)=J'$ is not one-to-one there exist two points x and y of J such that $T(x)=T(y)$. Express $J=\alpha+\beta$, where α and β are arcs such that $\alpha\cdot\beta=x+y$. There exist in α distinct points p and q whose images are the endpoints of $T(\alpha)$. (In case $T(\alpha)$ is degenerate these points may be x and y.) Let the points be so named that $\alpha=xp+pq+qy$, where any two of the arcs on the right have at most a common endpoint. Now the endpoints of $T(\alpha)$ lie in its subarcs $T(xp)$ and $T(qy)$, which have a common point. Consequently, $T(xp+qy)=T(xp)+T(qy)=T(\alpha)$. Thus the arc $\gamma=px+\beta+qy$ is such that $T(\gamma)=J'$, contrary to the hypothesis that J' is not an arc.

(3.3) *If $T(A)=B$ is arc-preserving, where A is cyclic, and if there exists a simple closed curve J in A such that $T(J)=J'$ is not an arc, then T is one-to-one on A.*

PROOF. Since J' is not an arc, it follows from (3.2) that $T(J)=J'$ is topological. Let z be any point of J and suppose $z\neq T^{-1}T(z)$. Then there exists a point z_1 in $A-J$ such that $T(z_1)=T(z)$. Let cz_1d be an arc in A spanning[6] J, and suppose $d\neq z$. Write J as the sum of two simple arcs α and β having precisely the points z and d in common.

[6] An arc axb is said to *span* a point set M provided $M\cdot axb=a+b$.

Now $T(z)=T(z_1)$ and T is topological on α. Consequently, since $T(z_1d+\alpha)$ is an arc, we have $T(z_1d)$ contains $T(\alpha)$. Therefore, $T(z_1d+\beta)$ contains J', contrary to the fact that J' is not an arc. Hence for every point z of J we have $z=T^{-1}T(z)$.

Now let z be a point of $A-J$ and let α be an arc through z spanning J and dividing J into two arcs β and γ. From the above and (3.2) it follows that $T(\alpha+\beta)$ is a simple closed curve. Consequently, $z=T^{-1}T(z)$ and, therefore, T is one-to-one on A.

(3.4) *Under the hypotheses of* (3.3) *either of the following conditions suffices to make T topological*: (a) *A is strongly arcwise connected, or* (b) *T sends trees into compact sets.*

PROOF. That (a) suffices is immediate from (2.5). To show that (b) suffices we need only establish the continuity of T. Assume T is not continuous. Then there exists a sequence of points $\{x_n\}$ converging to a point x of A such that either the sequence $T(x_n)=x_n'$ converges to a point $y'\neq T(x)$ or the set $T(x_n)$ has no limit point. Now there exists a tree t containing infinitely many of the x_n, but not $y=T^{-1}(y')$ if y' exists. It follows in either case that $T(t)$ is not compact.[7] This contradiction completes the proof.

(3.5) *Let J be any simple closed curve in the cyclic continuum A and let $T(A)=B$ be arc-preserving but not topological. Then either of the following conditions suffices to make $T(J)=J'$ a free arc*[7] *of B*: (a) *A is strongly arcwise connected, or* (b) *the image of each tree in A is a locally connected continuum.*

PROOF. From (3.2) we see that $T(J)=J'$ is topological or J' is an arc. If $T(J)=J'$ is topological, then by (3.4) we see that $T(A)=B$ is topological and the theorem is established. Hence assume that J' is an arc $a'x'b'$ of B which is not a free arc of B. To obtain the desired contradiction we first establish the following assertion:

(i) *There exists an arc uv in A such that $T(uv)$ contains a nondegenerate subarc $u'v'$ having exactly the point v' in common with J', where v' is an interior point of the arc J'.*

To prove (i) we observe that since J' is not a free arc of B there must exist a sequence of points x_n' of $B-J'$ converging to an interior point x' of J'. If A is strongly arcwise connected there exists an arc N in A intersecting infinitely many of the sets $T^{-1}(x_n')$. Then $T(N)$ contains a nondegenerate subarc $u'v'$ satisfying the conditions of (i) and

[7] An arc α is a *free* arc of M provided α spans $\overline{M-\alpha}$.

we immediately obtain the arc uv as a subarc of N joining a point of $T^{-1}(u')$ to a point of $T^{-1}(v')$. This proves (i) under our first hypothesis. Assume next that (b) holds. Then there exists a tree t in A intersecting $T^{-1}(x')$ and infinitely many of the sets $T^{-1}(x_n')$. Thus $T(t)$ is a locally connected continuum in B containing x' and infinitely many of the points x_n'. Hence $T(t)$ is locally arcwise connected and thus contains an arc $u'v'$ satisfying (i). The arc uv is then obtained as any arc in t joining a point of $T^{-1}(u')$ to a point of $T^{-1}(v')$. This completes the proof of (i).

Now since A is cyclic there exists an arc H in A intersecting $T^{-1}(u')$ and having its endpoints in $T^{-1}(a')$ and $T^{-1}(b')$. It follows at once from (3.1) that $T(H)$ contains J'. Now $T(H)$ contains both u' and v' and hence a subarc joining these points. Since $T(H)$ is an arc, this subarc must contain either a' or b', hence we assume that it contains a' and denote it by $u'a'v'$. From (3.1) it follows that if M is any arc in A having its endpoints in $T^{-1}(u')$ and $T^{-1}(v')$ then $T(M)$ contains $u'a'v'$. But this tells us at once that the image of the arc uv given by (i) must contain a simple closed curve. This contradiction completes the proof.

4. **Principal theorem.** We shall now prove our principal theorem.

(4.1) *Let $T(A)=B$ be arc-preserving, where A is a cyclic locally connected continuum and B is not an arc. Then T is topological if either A is strongly arcwise connected or T is tree-preserving.*

Proof. Assume T is not topological. Then by (3.5) the image of every simple closed curve J of A is a free arc of B and, consequently, every two points of B lie on a free arc of B. Thus to show that B is a simple closed curve it is certainly sufficient to show that B is a locally connected continuum. This follows at once from (2.1) and (2.4) if A is strongly arcwise connected. Hence we need only establish it in the case where T is tree-preserving. That B is compact follows at once from this condition since every convergent sequence of points in A lies on a tree in A and the image of this tree is a tree. Assuming B not locally connected we can find two points a and b in B at which B fails to be locally connected. Let axb be a free arc in B and d a point of B not on this free arc. Then there exists a free arc dx in B where x is an interior point of axb. Thus either a or b is interior to the free arc dx of B which is impossible since B is not locally connected at either a or b. Hence we have established the fact that under either of our hypotheses B must be a simple closed curve.

We show first that if z' is any point of B there exists an arc β of A

such that z' is an interior point of $T(\beta)$. Since B is a simple closed curve, it contains an arc $p'z'q'$ having z' as an interior point. Let $\{p_n'\}$ be a sequence of points in the arc $p'z'$ converging monotonically to z'. Let t be a tree in A intersecting infinitely many of the sets $T^{-1}(p_n')$. If T is tree-preserving then $T(t)$ is an arc $t'=a'b'$. There exists in t an arc α intersecting both $T^{-1}(a')$ and $T^{-1}(b')$. Since $T(\alpha)=t'$, it follows that $T(\alpha)$ must contain a subarc $\gamma_p=z'x'$ of $z'p'$. If A is strongly arcwise connected the same result may be obtained by taking α as an arc in A which intersects infinitely many of the sets $T^{-1}(p_n')$. Likewise we obtain an arc $\gamma_q=z'y'$ which is a subarc of both $z'q'$ and the image of an arc of A. Let β be an arc in A intersecting $T^{-1}(x')$, $T^{-1}(y')$, and $T^{-1}(z')$. By virtue of (3.1), $T(\beta)$ contains $\gamma_p+\gamma_q$, an arc which has z' as an interior point.

It follows from the above and the Heine-Borel theorem that there exists a finite number of arcs $\alpha_1, \alpha_2, \cdots, \alpha_n$ $(n\geqq 2)$ in A such that $\sum_{i=1}^n T(\alpha_i)=\sum\alpha_i'=B$. Moreover, the α_i may be so selected and named that $\alpha_i'\cdot\alpha_k'$ is empty except for $k=i-1, i, i+1$ $(n+1=1)$. Let p_i' and q_i' be the endpoints of α_i', and assume them so named that p_{i+1}' is a point of α_i' while q_{i+1}' is not. Select in A an arc $\beta_1=p_1q_1+q_1q_2$, where the points p_1, q_1, and q_2 are in $T^{-1}(p_1')$, $T^{-1}(q_1')$, and $T^{-1}(q_2')$ respectively. By (3.1), $T(p_1q_2)$ contains α_1', and consequently a point p_2 of $T^{-1}(p_2')$. The subarc p_2q_2 of β_1 has an image which contains α_2', by (3.1). Therefore $T(\beta_i)$ contains $\alpha_1'+\alpha_2'$. To complete the induction assume that an arc β_r has been obtained in A such that $T(\beta_r)$ contains $\alpha_1'+\alpha_2'+\cdots+\alpha_{r+1}'$. Select in A an arc $\beta_{r+1}=p_1q_{r+1}+q_{r+1}q_{r+2}$, where the points p_1, q_{r+1}, and q_{r+2} are in $T^{-1}(p_1)$, $T^{-1}(q_{r+1})$, and $T^{-1}(q_{r+2})$ respectively. By applying (3.1) as above it follows that $T(p_1q_{r+1})$ contains $\alpha_1+\cdots+\alpha_{r+1}$, and, finally, that $T(\beta_{r+1})$ contains $\alpha_1+\cdots+\alpha_{r+2}$. Therefore, by induction, there exists an arc β_{n-1} in A such that $T(\beta_{n-1})=B$, which, as a consequence of our supposition, is a simple closed curve. This contradiction completes the proof of the theorem.

(4.11) COROLLARY. *Let A be a locally connected continuum having no local separating points. If $T(A)=B$ is arc-preserving, then either B is an arc or T is topological.*

PROOF. Since A has no local separating point it can contain no cut point. Moreover, A is strongly arcwise connected, since every closed and totally disconnected set of A is contained in an arc.[8] Consequently, A satisfies the hypotheses of (4.1).

[8] See G. T. Whyburn, *On disconnected sets*, Fundamenta Mathematicae, vol. 18 (1931), pp. 48–60.

The following theorem is an immediate consequence of (4.1) and (2.6):

(4.2) *The class of cyclic strongly arcwise connected continua consists exactly of all cyclic locally connected continua A such that every arc-preserving transformation $T(A)=B$, where B is not an arc, is topological.*

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A NOTE ON SUBGEOMETRIES OF PROJECTIVE GEOMETRY AS THE THEORIES OF TENSORS[1]

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Klein's viewpoint (A) of a geometry as the invariant theory of a transformation group, as formulated in the Erlanger Programm in 1870,[2] has played an important part in the study of geometry during the past half century. A number of explicit utilizations of this viewpoint in invariant aspects of algebraic geometry have been made.[3] In the last decade the viewpoint (B) of a geometry as the theory of a tensor has received considerable theoretical discussion and utilization in connection with the new differential geometries.[4] While the adjunction argument, whereby subgeometries of projective geometry result from the latter by holding certain forms latent, has had considerable use,[5] and is closely related to tensor algebra, there seems to have been no explicit treatment of algebraic invariants for subgeometries of projective geometry from the viewpoint (B) with the use of tensor algebra. To indicate how this might be done is the purpose of this paper. The material here is largely an application and continuation of the basic paper by Cramlet.[6]

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[1] Presented to the Society, April 27, 1940.

[2] F. Klein, *Gesammelte Mathematische Abhandlungen*, Berlin, 1921, vol. 1, p. 460.

[3] C. C. MacDuffee, *Euclidean invariants of second degree curves*, American Mathematical Monthly, vol. 33 (1926), pp. 243–252; *Covariants of r-parameter groups*, Transactions of this Society, vol. 39 (1933).

[4] J. A. Schouten and J. Haantjes, *On the theory of the geometric object*, Proceedings of the London Mathematical Society, vol. 42 (1937), pp. 356–376.

[5] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1939, pp. 254–258; H. W. Turnbull, *The Theory of Determinants, Matrices, and Invariants*, Blackie and Son, 1929, chap. 21.

[6] C. M. Cramlet, *The derivation of algebraic invariants by tensor algebra*, this Bulletin, vol. 34 (1928), pp. 334–342.