ON THE SIMULTANEOUS APPROXIMATION
OF TWO REAL NUMBERS

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If $\xi_1, \xi_2, \ldots, \xi_n$ are any real numbers and $t$ is a positive integer, then it is well known that integers $a_1, a_2, \ldots, a_n, b$ can be found, such that $0 < b \leq t^n$ and

$$|b\xi_k - a_k| < \frac{1}{t}, \quad k = 1, 2, \ldots, n.$$  

The proof is briefly the following.\(^2\) Consider the $t^n+1$ points $(r\xi_1, r\xi_2, \ldots, r\xi_n)$, where $r = 0, 1, \ldots, t^n$. Reduce mod 1 to congruent points in the unit cube $(0 \leq x_1 < 1, \ldots, 0 \leq x_n < 1)$. If this cube is divided into $t^n$ cubes of edge $1/t$ (including the lower boundaries), then at least one of these small cubes must contain two of the reduced points, say those with $r=r'$ and $r=r''$. With $b = |r' - r''|$ and suitable $a$'s, we evidently satisfy the required inequalities.

For $n = 1$, the inequality can be sharpened to

$$|b\xi - a| \leq \frac{1}{t+1},$$

$b$ satisfying the condition $0 < b \leq t$. For if we consider the points $r\xi$ ($r = 0, 1, \ldots, t$), and mark the points in the interval $0 \leq x \leq 1$ which are congruent to them mod 1, we have at least $t+2$ points marked, since corresponding to $r=0$ we mark both 0 and 1. Some two of the marked points must lie within a distance $1/(t+1)$ from each other, so that the desired conclusion follows. This is the best result, as the example $\xi = 1/(t+1)$ shows.

The present note solves the corresponding problem for $n = 2$. For larger values of $n$ the problem appears more difficult.

THEOREM. If $\xi_1$ and $\xi_2$ are any real numbers, and $s$ is a positive integer, then integers $a_1, a_2, b$ can be found, such that $0 < b \leq s$, and

$$|b\xi_k - a_k| \leq \max\left(\frac{[s^{1/2}]}{s+1}, \frac{1}{[s^{1/2}]+1}\right), \quad k = 1, 2.$$  

For every $s$, values of $\xi_1$ and $\xi_2$ can be found for which the inequalities could not both be satisfied if the equality sign were omitted.

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1 Presented to the Society, November 23, 1940.

2 The method used in this proof (\textit{Schubfachprinzip} or "pigeonhole principle") was first used by Dirichlet in connection with a similar problem. We sketch the proof here in order to compare it with the proof of the theorem below, which also uses that method.
The inequalities may also be written

\[ |b\xi_k - a_k| \leq \begin{cases} \frac{t}{(s+1)} & \text{for } t^2 - 1 \leq s \leq t(t + 1) - 1, \\ \frac{1}{t + 1} & \text{for } t(t + 1) - 1 \leq s \leq (t + 1)^2 - 1. \end{cases} \]

It will be noted that in some intervals the bound does not decrease as \( s \) increases.

We show first that the theorem is the best possible. We shall think of the inequalities in the form just given. If \( s < (t + 1)^2 \), then it is evident that \( \xi_1 = 1/(t+1) \), \( \xi_2 = 1/(t+1)^2 \) are a pair of real numbers which cannot be approximated simultaneously with an error less than \( 1/(t+1) \); this settles the second case. For the first case, consider the pair of real numbers \( \xi_1 = 1/(s+1) \), \( \xi_2 = t/(s+1) \). We are to show that not both errors can be made less than \( 1/(s+1) \). We note first that \( b\xi_1 \) and \( b\xi_2 \) differ from integers by the same amount as \( (s+1-b)\xi_1 \) and \( (s+1-b)\xi_2 \); hence we may suppose that \( b \leq (s+1)/2 \), and therefore \( 0 < b\xi_1 \leq 1/2 \). In order to make \( |b\xi_1 - a_1| < t/(s+1) \), we must have \( 0 < b < t \). Then \( 0 < b\xi_2 < 1 \). Since \( b\xi_2 \geq \xi_2 = t/(s+1) \) and \( 1 - b\xi_2 \geq 1 - (t-1)\xi_2 = 1 - (t-1)t/(s+1) \geq t/(s+1) \), we see that the inequality \( |b\xi_2 - a_2| < t/(s+1) \) cannot be satisfied.

The theorem evidently follows from the lemma below, by putting \( t = \lfloor s^{1/2} \rfloor \).

**Lemma.** Let \( s \) and \( t \) be positive integers with \( s \geq t \). If \( \xi_1 \) and \( \xi_2 \) are any real numbers, then integers \( a_1 \), \( a_2 \), \( b \) can be found, such that \( 0 < b \leq s \), and

\[ |b\xi_1 - a_1| \leq t/(s + 1), \quad |b\xi_2 - a_2| \leq 1/(t + 1). \]

**Proof.** Consider the points \( (r\xi_1, r\xi_2) \) with \( r = 0, 1, \ldots, s \). Mark all the points congruent to these mod 1 which fall in the rectangle \( 0 \leq x_1 \leq t, 0 \leq x_2 < 1 \). There are \((s+1)t\) points to be marked with \( x_1 < t \); and in addition, the point \((t, 0)\) is marked, corresponding to \( r = 0 \). If we divide our rectangle into \( s+1 \) rectangles of width \( t/(s+1) \) (closed except at the top) by means of vertical lines, then at least one of them contains more than \( t \) points, all corresponding to different values of \( r \). The corresponding values of \( x_2 \) are \( t+1 \) or more numbers, some two of which differ mod 1 by not more than \( 1/(t+1) \). Thus we find two points \((r'\xi_1, r'\xi_2)\) and \((r''\xi_1, r''\xi_2)\), whose horizontal distance mod 1 does not exceed \( t/(s+1) \) and whose vertical distance mod 1 does not exceed \( 1/(t+1) \). Putting \( b = |r'-r''| \) gives the required result.

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