CREMONA INVOLUTIONS DETERMINED BY TWO LINE CONGRUENCES

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1. Introduction. Recently, in this Bulletin,[1] we discussed a multiple null-correspondence formed by two line congruences. The *first congruence* consisted in the lines intersecting a fixed twisted curve of order m and also its $(m-1)$-secant d, and the *second congruence* consisted in the lines intersecting another twisted curve δ_n', of order n, and its $(n-1)$-secant d'.

When $m=2$, the first curve is a conic c_2 having one point on d; a series of space Cremona involutions may now be defined as follows:

A generic point P determines a ray ρ_1 of the first congruence and a ray ρ' of the second congruence. In the null-plane of P, formed by ρ_1 and ρ', lies another ray ρ_2 of the first congruence which intersects ρ' in P', the correspondent of P in the involution.

2. The defining curves in general position. Let the equations of d be $x_1=0$, $x_2=0$; and those of d' be $x_3=0$, $x_4=0$. Let the parametric equations of c_2 and of δ_n' be, respectively,

$$x_1 = \lambda^2, \qquad x_1 = f_n(s, t),$$

$$x_2 = \lambda(\lambda - \mu), \qquad x_2 = F_n(s, t),$$

$$x_3 = \lambda(\lambda - \mu), \qquad x_3 = \prod_1^{n-1} (t_i s - s_i t)(as + bt),$$

$$x_4 = \mu^2, \qquad x_4 = \prod_1^{n-1} (t_i s - s_i t)(cs + dt),$$

where s_i, t_i are the values of the parameters of δ_n' at the $n-1$ points on d'.

Then the equations of the involution are

$$x_1' = (x_2 - x_3)(x_1F - x_2f)^2,$$

$$x_2' = (x_1F - x_2f)L,$$

$$x_3' = x_3\{(x_2 - x_3)(x_1F - x_2f)(F - kHx_2) + (kHx_1 - f)L\},$$

$$x_4' = x_4\{(x_2 - x_3)(x_1F - x_2f)(F - kHx_2) + (kHx_1 - f)L\},$$

where $f \equiv f_n(dx_3 - bx_4, \quad ax_4 - cx_3), \quad k \equiv (ad - bc), \quad F \equiv F_n(dx_3 - bx_4,$

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[1] E. J. Purcell, *A multiple null-correspondence and a space Cremona involution of order $2n-1$*, this Bulletin, vol. 46 (1940), pp. 339–344.

$ax_4 - cx_3$), $H \equiv \prod_1^{n-1} \{(s_i c + t_i d)x_3 - (s_i a + t_i b)x_4\}$, $L \equiv (x_2 - x_3)(x_1 F - x_2 f)$
$- (x_1 - x_2)(kHx_1 x_3 - kHx_2 x_3 + Fx_3 - Fx_1 - fx_3 + fx_4) - x_1 x_4 (F - kHx_2)$.

The order of the involution is $2n+3$, where n is any integer. The fundamental system and its principal images follow.

Line d is a simple F-line of simple contact. The fixed tangent plane to all the homaloids is $x_1 = 0$, which is the plane through d tangent to c_2. d is of the first species and its P-surface is a ruled surface, $x_1 F - x_2 f = 0$, of order $n+1$ formed by the rays of the second congruence which intersect d. Each such ray goes over into its point of intersection with d. On this P-surface, d is a simple directrix line, δ_n' is a simple directrix curve, and d' is an n-fold directrix line.

Line d' is a $2n$-fold F-line without contact. It is of the first species. Its P-surface is of order $2n+2$, whose equation is $(x_2 - x_3)(x_1 F - x_2 f)(F - kHx_2) + (kHx_1 - f)L = 0$. On this P-surface d is a simple line, d' is a $(2n-1)$-fold line, c_2 is a simple conic, and δ_n' is a double curve; by their intersections with these curves, all the F-lines of second species have more than $2n+2$ points on the P-surface and therefore lie entirely on this surface.

Conic c_2 is a simple F-conic without contact. It is of the first species and has for P-surface the ruled surface of order $2n+2$ formed by the rays of the second congruence which intersect c_2. Each such ray goes over into its point of intersection with c_2. On this P-surface c_2 is a simple directrix curve, δ_n' is a double directrix curve, and d' is a $2n$-fold directrix curve.

Curve δ_n' is a double F-curve without contact. It is of the first species and has for P-surface $(x_2 - x_3)(x_1 F - x_2 f)F - fL = 0$, which is of order $2n+2$. On this P-surface d is a simple line, d' is a $2n$-fold line, c_2 is a simple conic, and δ_n' is a simple curve. Since $x_1 = 0$ is tangent to this surface along d, all the F-lines of second species lie on this P-surface.

The first is a $(1, 2)$ congruence and the second is a $(1, n)$ congruence. There are thus $2n+1$ common rays of both congruences. Each is a simple F-line of simple contact and is of the second species. The contact along these lines is torsal—that is, while all the homaloids have the same fixed tangent plane at a particular point on the line, the fixed tangent plane turns as this point moves along the line.

In the plane of c_2 are n rays of the second congruence. Each is a simple F-line without contact and is of the second species.

The fixed tangent plane, through d, to all the homaloids contains n rays of the second congruence. Each is a simple F-line without contact and is of the second species.

The locus of invariant points is a surface Σ of order $n+3$. Its equa-

tion is $x_1L - x_2(x_2 - x_3)(x_1F - x_2f) = 0$. On this surface d is a double line, d' is an n-fold line, c_2 is a simple conic, and δ_n' is a simple curve.

An alternate definition of the same involution employs this surface Σ. A generic point P determines a ray ρ' intersecting d' once and δ_n' once. Ray ρ' has with Σ two intersections, α and β, not on d' or δ_n'. We define P', the correspondent of P in the involution, to be the harmonic conjugate of P with respect to α and β.

3. **Lower order for d' in special position.** Any plane through d' intersects δ_n' in one point Z not on d'.

If d' lies in the plane of c_2, this plane will contain such a point Z. Any line in the plane of c_2 passing through Z is a ray of the second congruence. Each such line intersects d' once, δ_n' once, and c_2 twice. It has $2n+4$ intersections with every homaloid and therefore lies on every homaloid. The totality of these lines is the plane of c_2 which, being part of every homaloid, reduces the order of the involution by one.

The fundamental system is modified and the locus of invariant points is a surface of order $n+2$ on which d' is an $(n-1)$-fold line, d is a double line, and δ_n' is a simple curve.

If the fixed tangent plane along d to all the homaloids contains d', every line on it passing through Z will lie on every homaloid. This plane will factor from the equations of the involution and their order will be one lower than in the general case.

Here the locus of invariant points is a surface, Σ, of order $n+2$ on which d' is an $(n-1)$-fold line, δ_n' is a simple curve, c_2 is a simple conic; but d does not lie on Σ.

4. **The defining curves on one quadric surface.** Let d, d', c_2, and δ_n' lie on the same quadric surface, d and d' being generators of the same system. Since c_2 intersects d once, it intersects d' once. δ_n' has $n-1$ points Δ_i' on d' and therefore $n-1$ points Δ_j on d. It follows that c_2 and δ_n' intersect in n points. Every generator of the other system intersects d, d', c_2, and δ_n', each once and thus has $1+2n+1+2 = 2n+4$ intersections with every homaloid. Their locus is the quadric surface which is thus a part of every homaloid and factors from the equations of the involution, reducing the order by two. The involution in this case is of order $2n+1$.

If the quadric surface is $x_1x_3 - x_2x_4 = 0$; d is $x_1 = 0$, $x_2 = 0$; d' is $x_3 = 0$, $x_4 = 0$; and the parametric equations of c_2 and δ_n' are, respectively,

$$x_1 = \lambda\mu, \qquad x_1 = \prod_n^{2n-2} (t_j s - s_j t) K(cs + dt),$$

$$x_2 = \lambda^2, \qquad x_2 = \prod_{n}^{2n-2} (t_j s - s_j t) K(as + bt),$$

$$x_3 = \lambda\mu, \qquad x_3 = \prod_{1}^{n-1} (t_i s - s_i t)(as + bt),$$

$$x_4 = \mu^2, \qquad x_4 = \prod_{1}^{n-1} (t_i s - s_i t)(cs + dt),$$

then the equations of the involution are

$$x_1' = KW(Hx_1 - KWx_4)(x_1 x_3 - x_2 x_4),$$
$$x_2' = K^2 W^2(x_1 - x_3)(x_1 x_3 - x_2 x_4),$$
$$x_3' = x_3(Hx_1 - KWx_4)(KWx_1 - Hx_2),$$
$$x_4' = x_4(Hx_1 - KWx_4)(KWx_1 - Hx_2),$$

where K is a constant, H is as in §2, and

$$W \equiv \prod_{n}^{2n-2} \{(s_j c + t_j d)x_3 - (s_j a + t_j b)x_4\}.$$

In this involution, d is a simple F-line without contact. It is of the first species and its P-surface is $x_1 x_3 - x_2 x_4 = 0$.

d' is a $(2n-1)$-fold F-line without contact. It is of the first species and its P-surface is $KWx_1 - Hx_2 = 0$.

δ_n' is a simple F-curve of order n, of the first species, whose P-surface is $x_1 = 0$.

Δ_j, $(j=n, n+1, \cdots, 2n-2)$, the intersections of δ_n' and d, are isolated double F-points. Their P-surfaces are, respectively, $(s_j c + t_j d)x_3 - (s_j a + t_j b)x_4 = 0$.

$D_1, D_2, \cdots, D_n$, the n intersections of c_2 and δ_n', are isolated double F-points. The P-surface of each D_i is the plane of that D_i and d'.

Point $R(0100)$, the intersection of c_2 and d', is a $2n$-fold isolated F-point. Its P-surface is $Hx_1 - KWx_4 = 0$.

The F-curves of second species are as follows:

Lines $R\Delta_j$, $(j=n, n+1, \cdots, 2n-2)$, are simple F-lines of simple contact.

Lines RD_i, $(i=1, 2, \cdots, n)$, are simple F-lines without contact.

Through each D_i there passes a single line intersecting both d and d'. Each is a simple F-line without contact.

The plane of R and d, $x_1 = 0$, intersects the quadric surface in two lines, d and RS. $x_1 = 0$ intersects δ_n' in n points, $n-1$ of which are the Δ_j, $(j=n, n+1, \cdots, 2n-2)$, on d, and the remaining point is

on RS. Thus RS has $2n$ points of intersection with every homaloid at R, one at δ_n', and one at d. Line RS is a simple F-line without contact.

Since $Q(1000)$ lies on d', it is a $(2n-1)$-fold point. But at Q, $n-1$ sheets of every homaloid have contact with the $n-1$ planes, $W=0$, through Q. The planes $W=0$ intersect d in Δ_j, $(j=n, n+1, \cdots, 2n-2)$. Each line $Q\Delta_j$ has $2n$ points of intersection with every homaloid at Q and two at Δ_j. Lines $Q\Delta_j$ are simple F-lines without contact.

The locus of invariant points is Σ, a surface whose equation is $Kx_1W(x_1-x_3)-x_2(Hx_1-KWx_4)=0$. Σ is of order $n+1$ and contains d' as $(n-1)$-fold line without contact. d is a simple line on Σ, but neither δ_n' nor c_2 is on this surface. The alternate definition for this involution is: a generic point P determines a ray ρ' intersecting d' once and δ_n' once. Ray ρ' intersects Σ $n-1$ times on d', and in two other points, α and β, not on d'. P', the correspondent on P in the involution, is the harmonic conjugate of P with respect to α and β.

5. The second congruence consists in the bisecants of a space cubic curve.

We now consider the involution of order 9 that results when the second congruence consists in the bisecants of a twisted cubic curve, q_3'.

Let d be $x_1=0$, $x_2=0$. Let c_2 be $x_1=\lambda^2$, $x_2=\lambda\mu$, $x_3=\lambda^2$, $x_4=\mu^2$. Let the space cubic curve q_3' be $x_1=\sigma^3$, $x_2=\tau^3$, $x_3=\sigma\tau^2$, $x_4=\sigma^2\tau$. Then the equations of the involution are

$$x_1' = (x_1 - x_3)(C^2 - AB)^2,$$

$$x_2' = (C^2 - AB)N,$$

$$x_3' = (x_1 - x_3)A^2(C^2 - AB) - BCN,$$

$$x_4' = B^2N - (x_1 - x_3)AC(C^2 - AB),$$

where $A \equiv x_3^2 - x_2x_4$, $B \equiv x_4^2 - x_1x_3$, $C \equiv x_1x_2 - x_3x_4$, and $N \equiv x_1^3x_3^2 - x_1^2x_3^3 + x_1^2x_2x_3x_4 + x_1x_3^2x_4^2 + x_2x_4^3 - x_1^2x_2^3 - 2x_2^2x_3^2x_4 + x_2^3x_4^2 - x_1x_2x_4^3 - x_2^2x_4^3 - x_1x_2x_3^3 + 3x_1x_2^2x_3x_4 + x_1x_4^4 - 2x_1^2x_3x_4^2$.

The order of this involution is 9.

In this involution, d is a simple F-line of simple contact. It is of the first species and its P-surface, $C^2-AB=0$, is a ruled surface of order 4 consisting in the rays of the second congruence that intersect d. Each such ray goes over into its point of intersection with d.

c_2 is a simple F-conic of the first species whose P-surface is ruled and of order 8, being formed by the rays of the second congruence that intersect c_2.

q_3' is a fourfold F-cubic without contact. It is of the first species and its P-surface is of order 8.

Since the first congruence is of order 1 and class 2 and the second congruence is of order 1 and class 3, there are 7 common rays of both congruences. Each such ray intersects every homaloid 4 times at each of its two intersections with q_3', once on d, and once on c_2. Thus the seven common rays of both congruences lie on every homaloid. They are simple F-lines of simple contact.

The fixed tangent plane, $x_1 = 0$, through d intersects q_3' in three points in general. The three lines joining these points two and two are rays of the second congruence lying in $x_1 = 0$. Each such ray intersects every homaloid 4 times at each of its two points on q_3' and twice on d. These three lines are simple F-lines without contact and are of the second species.

The plane of c_2 in general cuts q_3' in three points. There are thus three rays of the second congruence which cut c_2 twice. They are simple F-lines without contact and are of the second species.

Σ, the locus of invariant points, is $x_1 N - x_2(x_1 - x_3)(C^2 - AB) = 0$. This surface is of order 6 and contains q_3' as a double cubic curve. d is a double line and c_2 is a simple conic on Σ.

Using this surface, we may define our involution: a generic point P determines a bisecant ρ' of the cubic curve q_3'. Ray ρ' intersects Σ in two points, α, β, not on q_3'. P', the correspondent of P, is the harmonic conjugate of P with respect to α and β.

When d, c_2, and q_3' lie on the same quadric surface, with d intersecting q_3' in one point, all the generators of the other system drop away and the order of the involution is lower by two. A discussion similar to that in §5 applies to this involution of order 7.

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