RATIONAL APPROXIMATIONS TO IRRATIONALS

ALEXANDER OPPENHEIM

It is well known that if p/q is a convergent to the irrational number x, then $|x-p/q|<1/q^2$. The immediate converse is of course false but I have not seen in the literature[1] any statement of the converse which is given below.

THEOREM 1. *If p and q are coprime, $q>0$, and if $|x-p/q|<1/q^2$, then necessarily p/q is one of the three (irreducible) fractions*

$$p'/q', \qquad (p'+p'')/(q'+q''), \qquad (p'-p'')/(q'-q''),$$

where p''/q'', p'/q' are two consecutive convergents to the irrational x. One at least of the two fractions $(p'+\epsilon p'')/(q'+\epsilon q'')$ where $\epsilon=\pm 1$ satisfies the inequality.

In other words if the inequality is satisfied, then

$$p/q = [a_1, a_2, \cdots, a_{n-1}, a_n + c], \qquad c = 0, \pm 1,$$

where $[a_1, a_2, \cdots, a_r, \cdots]=x$ is the infinite simple continued fraction for x, so that the a_i are integers, $a_i \geqq 1$ $(i\geqq 2)$.

Suppose that $x-p/q=\epsilon\theta/q^2$, $0<\theta<1$, $\epsilon=\pm 1$. Let

$$p/q = [b_1, b_2, \cdots, b_m], \qquad p'/q' = [b_1, b_2, \cdots, b_{m-1}],$$

where m (which we can choose to be odd or even) is taken so that $(-1)^{m-1}=\epsilon$. Defining y by the equation

$$x = [b_1, b_2, \cdots, b_m, y] = (yp+p')/(yq+q'),$$

we obtain $\epsilon\theta=q^2(x-p/q)=(p'q-pq')q/(yq+q')$; so that, since $p'q-pq'=(-1)^{m-1}=\epsilon$, $y+q'/q=1/\theta$.

Since $1/\theta>1$ and $q'/q<1$ it follows that $y>0$.

If $y>1$, then $y=[b_{m+1}, b_{m+2}, \cdots]$ $(b_{m+1}\geqq 1, \cdots)$, and so $x=[b_1, b_2, \cdots, b_m, b_{m+1}, \cdots]$, which, since the infinite simple continued fraction is unique, shows that $p/q=[b_1, \cdots, b_m]$ is the mth convergent to x. If however $y<1$, then $1/y=[c, b_{m+1}, b_{m+2}, \cdots]$ with $c\geqq 1$. But $q/q'=[b_m, b_{m-1}, \cdots, b_2]$ and therefore one of c and b_m must be unity for, if not, then $1/y>2$, $q/q'>2$, $y+q'/q<1<1/\theta$.

[1] Editor's note. In the meantime, R. M. Robinson has proved similar results in the Duke Mathematical Journal, vol. 7 (1940), pp. 354–359. Also the first part of Theorem 1 was observed by P. Fatou, Comptes Rendus de l'Académie des Sciences, Paris, vol. 139 (1904), pp. 1019–1021.

Hence $x=[b_1, \cdots, b_{m-1}, b_m+c, b_{m+1}, \cdots]$ and $b_i=a_i$ $(i\neq m)$, $b_m+c=a_m$. Thus $p/q=[a_1, a_2, \cdots, a_{m-1}, b_m]$ where $b_m=1$ or a_m-1. Consequently

$$p/q = (p_{m-1} + p_{m-2})/(q_{m-1} + q_{m-2}), \qquad b_m = 1,$$

or

$$p/q = (p_m - p_{m-1})/(q_m - q_{m-1}), \qquad b_m = a_m - 1.$$

The first part of the theorem is proved.

Now let $p/q=(p_n+\epsilon p_{n-1})/(q_n+\epsilon q_{n-1})$, $\epsilon=\pm 1$, p_n/q_n being the nth convergent to $x=[a_1, \cdots, a_n, x']=(x'p_n+p_{n-1})/(x'q_n+q_{n-1})$. Then

$$q^2 | x - p/q | = | x'\epsilon - 1 | (q_n + \epsilon q_{n-1})/(x'q_n + q_{n-1}),$$

which, since $|\epsilon|=1$, $x'>1$, is less than unity if and only if

$$\epsilon(x' - q_n/q_{n-1}) < 2.$$

But this inequality is certainly satisfied when ϵ has the sign opposite to the sign of $x'-q_n/q_{n-1}$. The second part of the theorem follows.

Irreducible fractions p/q can be divided into three classes $[o/e]$, $[e/o]$, $[o/o]$ in which o and e denote odd and even integers respectively.

Since $p_nq_{n-1}-p_{n-1}q_n=\pm 1$ it is clear that consecutive convergents p_{n-1}/q_{n-1}, p_n/q_n belong to two different classes and hence that $(p_n+\epsilon p_{n-1})/(q_n+\epsilon q_{n-1})$ where $\epsilon=\pm 1$ must belong to the remaining class of irreducible fractions. It follows from Theorem 1 that *for any irrational x infinitely many fractions of each class exist such that* $|x-p/q|<1/q^2$.

Theorem 1 in fact determines all such fractions.

This result is due to Scott[2] who used the geometric properties of elliptic modular transformations. Scott also showed that the result is the best possible: *for a given class and a fixed k, $0<k<1$, irrationals exist, dense everywhere on the real axis, such that the inequality $|x-p/q|<k/q^2$ is satisfied by only a finite number of fractions in the given class.*

To prove the last statement it will be enough to show that, *if $x=[a_1, a_2, \cdots, a_n, \cdots]$ where the a_n are even integers not less than* $2E+1$, where $E>1$, then for every fraction of type $[o/o]$,

$$\theta = q^2 | x - p/q | > 1 - 1/E.$$

If $\theta>1$, there is nothing to prove. If $\theta<1$, it follows from our theo-

[2] W. T. Scott, this Bulletin, vol. 46 (1940), pp. 124–129.

rem that (p/q being irreducible)

$$p = p_n + \epsilon p_{n-1}, \qquad q = q_n + \epsilon q_{n-1}, \qquad \epsilon = \pm 1,$$

for the convergents to x are all $[e/o]$ or $[o/e]$. Write $X=[a_{n+1}, a_{n+2}, \cdots]$, $Y=[a_n, a_{n-1}, \cdots, a_2]$. Then if $n \geqq 2$,

$$\theta = \frac{(Y+\epsilon)(X-\epsilon)}{XY+1} = 1 - \frac{2-\epsilon(X-Y)}{XY+1} > 1 - \frac{2+X+Y}{XY+1},$$

$$\begin{aligned} XY + 1 - E(2+X+Y) &= (X-E)(Y-E) - E^2 - 2E + 1 \\ &> (E+1)^2 - E^2 - 2E + 1 > 0, \end{aligned}$$

$$\theta > 1 - 1/E.$$

If $n=1$, then $p=p_1+1$, $q=q_1=1$, $\theta=1-[0, a_2, \cdots]>1-1/E$.

RAFFLES COLLEGE, SINGAPORE

MEASURABILITY AND DISTRIBUTIVITY IN THE THEORY OF LATTICES[1]

M. F. SMILEY

Introduction. Garrett Birkhoff[2] derived the following self-dual symmetric condition that a metric lattice be distributive:

$$(1) \quad \begin{aligned} 2[\mu(a \cup b \cup c) - \mu(a \cap b \cap c)] &= \mu(a \cup b) - \mu(a \cap b) + \mu(a \cup c) \\ &\quad - \mu(a \cap c) + \mu(b \cup c) - \mu(b \cap c). \end{aligned}$$

In a previous note[3] the author introduced and discussed a generalization of Carathéodory's notion of measurability[4] with respect to an outer measure function μ which applies to arbitrary lattices L. The μ-measurable elements form a subset $L(\mu)$ consisting of those elements $a \in L$ which satisfy

$$(2) \qquad \mu(a \cup b) + \mu(a \cap b) = \mu(a) + \mu(b)$$

for every $b \in L$. Closure properties of $L(\mu)$ were investigated. In par-

[1] Presented to the Society, January 1, 1941. The author wishes to express his gratitude to the referee for his valuable suggestions and comments.

[2] *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, p. 81. We shall adopt the notation and terminology of this work and shall indicate specific references to it by B.

[3] *A note on measure functions in a lattice*, this Bulletin, vol. 46 (1940), pp. 239–241. We shall indicate references to this paper by M.

[4] *Vorlesungen über Reelle Funktionen*, 2d edition, p. 246.