A FIXED-POINT THEOREM FOR TREES[1]

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By a *tree* we mean a compact (= bicompact) Hausdorff space which is acyclic in the sense that

(i) *if* $\mathfrak{U}$ *is a f.o.c. (= finite open covering) of a tree* T *then there is a f.o.c.* $\mathfrak{V}\subset\mathfrak{U}$ *such that the nerve* $N(\mathfrak{V})$ *is a combinatorial tree,*

and which is locally connected in the sense that

(ii) *if* $\mathfrak{U}$ *is a f.o.c. of* T *then there is a f.o.c.* $\mathfrak{V}\subset\mathfrak{U}$ *whose vertices are connected sets.*

It may be shown [**3**] that an acyclic continuous curve in the usual sense is a tree in our terminology. If q is a mapping which assigns to each point t of a topological space a *set* qt in a topological space, then we say that q is *continuous* provided that for each t and each neighborhood U of qt we can find an open set V containing t such that if t' is in V then qt' is in U. Our present purpose is to establish the following result:

(A) *Let* T *be a tree and let* q *be a continuous point-to-set mapping which assigns to each point* t *a continuum* qt *in* T. *Then there is a* $t_0\in T$ *such that* $t_0\in qt_0$.

The proof (which is divided into several lemmas) uses strongly a technique introduced by H. Hopf [**1**]. However the present note has been made self-contained.

(A_1) *The intersection of two continua of* T *is again a continuum.*

PROOF. Let B_1, B_2 be two continua such that $B_1\cdot B_2=C_1+C_2$ where the C_i are disjoint and closed. We can find disjoint open sets $D_i\supset C_i$. Let $t\in T-B_1\cdot B_2$. We can then find an open set V_t containing t and which does not meet both B_1 and B_2. The sets D_i together with the sets V_t can be reduced to a f.o.c. $\mathfrak{U}$ of T. Let $\mathfrak{V}\subset\mathfrak{U}$ be the f.o.c. described in (i). Let $\mathfrak{V}_i$ be those vertices of $\mathfrak{V}$ on B_i. It is easy to see that $N(\mathfrak{V}_i)$ is connected. If $c_j\in C_j$ we can find a chain of 1-cells E_i in $N(\mathfrak{V}_i)$ whose first vertex contains c_1 and whose last vertex contains c_2. Now we cannot have $E_i\subset D_1+D_2$ and E_i contains a vertex which is not on B_j. Hence $E_1\neq E_2$ and so $N(\mathfrak{V})$ is not a tree. This contradiction completes the proof.

[1] Presented to the Society, May 3, 1941.

(A$_2$) *Any f.o.c.* $\mathfrak{U}$ *of* T *contains a f.c.c.* $\mathfrak{F}\subset\mathfrak{U}$ *so that each* $F_i\in\mathfrak{F}$ *is connected and further* $N(\mathfrak{F})$ *is a combinatorial tree.*

PROOF. We can find a f.o.c. $\mathfrak{B}\subset\mathfrak{U}$ such that $N(\mathfrak{B})$ is a tree. By a lemma due to Čech [5, p. 180] we can find a f.c.c. $\mathfrak{F}'\subset\mathfrak{B}$ such that $\mathfrak{F}'$ and $\mathfrak{B}$ are combinatorially isomorphic. Let $\mathfrak{R}_i$ be the f.o.c. $(V_i, T-F_i')$. Using (ii) it is easy to see that there is a f.o.c. $\mathfrak{W}$ such that each W_i is connected and $\overline{\mathfrak{W}}\subset\mathfrak{R}_i$, for each i. Let i be fixed. If W_j meets F_i' then so does $\overline{W}_j$ and so is contained in V_i. Let Q_i be the union of all such W_j. Then the closure of this set has a component-wise decomposition, say $\overline{Q}_i=F_{i1}+F_{i2}+\cdots+F_{is_i}$. Let $\mathfrak{F}$ be the f.c.c. $\{F_{ij}\}$. It is clear that the elements of $\mathfrak{F}$ are connected and it is not hard to show that $\dim\mathfrak{F}\leqq 1$, that is, at most two elements of $\mathfrak{F}$ have a non-null intersection. If we have a chain

$$F_{i_1j_1}, F_{i_2j_2}, \cdots, F_{i_rj_r}, F_{i_1j_1}, \qquad r>2,$$

such that each set meets the following but such that there are no other intersections, then the sets $F_{i_1j_1}$ and $\sum_{s>1}F_{i_sj_s}$ are connected and therefore by (A$_1$) so is their meet, the set $F_{i_1j_1}\cdot F_{i_2j_2}+F_{i_1j_1}\cdot F_{i_rj_r}$. But then we would have $F_{i_1j_1}\cdot F_{i_2j_2}\cdot F_{i_rj_r}\neq 0$, a contradiction. It follows that $N(\mathfrak{F})$ is a tree.

(B) *Let* q *be a mapping which assigns to each continuum* K *in* T *a continuum* qK *in* T *such that if* $K_1\subset K_2$, *then* $qK_1\subset qK_2$. *If* $\mathfrak{F}=\{F_i\}$ *is a f.c.c. with connected sets such that* $N(\mathfrak{F})$ *is a tree then there is an* F_i *for which* $F_i\cdot qF_i\neq 0$.

PROOF. Let $N=N(\mathfrak{F})$ and suppose that the vertices of N are e_i. To each i we assign an i' so that $F_{i'}$ meets qF_i. We then have a mapping $e_i\to e_{i'}$ and since N is a tree it follows at once by a result due to Hopf [1, Lemma γ] that we can find an edge e_me_n which is contained in the chain joining $e_{m'}$ to $e_{n'}$.[2] We show that $F_k\cdot qF_k\neq 0$, $k=m$, or n. We have $F_m\cdot F_n\neq 0$ and by construction $F_{m'}\cdot qF_m\neq 0\neq F_{n'}\cdot qF_n$. Further

$$(*) \qquad F_{m'}, F_i, \cdots, F_m, F_n, F_j, \cdots, F_{n'}$$

is a simple chain of sets. Of course it may happen that F_n precedes F_m in (*) but this is of no importance. Let X be the union of all the sets in (*) from $F_{m'}$ up to and including F_m. Let Y be similarly defined for the other part of (*). Then X and Y are continua with $X\cdot Y=F_m\cdot F_n$.

[2] I am indebted to Professor S. Lefschetz for the remark that $e_i\to e_{i'}$ generates a chain-mapping (that is, a mapping permutable with the boundary operator) if we define for the image of e_me_n the chain joining e_m to e_n. Since N is acyclic it follows at once that there is a fixed element. This may replace the result of Hopf.

Also F_m+F_n is a continuum and so is $Z=qF_m+qF_n$. Clearly Z meets the end-vertices of (*). By (A_1) $Z\cdot(X+Y)$ is a continuum. Hence $Z\cdot X\cdot Y$ is not null. Thus $F_m\cdot F_n\cdot(qF_m+qF_n)\neq 0$ and this completes the proof of (B).

It is not hard to see that if q is a mapping of the type described in (A) then q satisfies the conditions in (B) if we define $qK=\sum qt$, $t\in K$, for each continuum K of T. The proof is quite similar to those for analogous results concerning single-valued mappings.

We now turn to a proof of (A). Suppose that no t is in qt. We can find a neighborhood R_t of t so that $\overline{R}_t$ does not meet qt. Let $V_t=T-\overline{R}_t$. Since $qt\subset V_t$ we can find a neighborhood S_t of t so that $t'\in S_t$ implies $qt'\subset V_t$. Let U_t be the meet of R_t and S_t. We cover T by a finite subcollection $\{U_i\}=\{U_{t_i}\}$ of the sets U_t. We can find a refinement $\mathfrak{F}$ of $\mathfrak{U}=\{U_i\}$ which satisfies the conditions in (B) in consequence of (A_2). By (B) we can find a set F in $\mathfrak{F}$ so that F meets qF. In other words we find a t in F such that F meets qt. Now F is in some U_i and hence qt is in the corresponding V_i. But since F does not meet the set V_i it cannot meet qt. This contradiction completes the proof.

A continuous transformation $fM=N$ is said to be *free* (Hopf [1]) provided there is a continuous transformation $gM\subset M$ such that $fgx\neq fx$ for each $x\in M$. The transformation f is *monotone* if the set $f^{-1}y$ is connected for each $y\in N$.

(C) *No continuum admits a free monotone transformation onto a tree.*

Proof. Let $fM=T$ be monotone and $gM\subset M$ be continuous. For each $t\in T$ we set $qt=fgf^{-1}t$. It is not hard to see that q is continuous and hence we may apply (A). But from $t\in qt$ it follows at once that there is an $x\in M$ with $fgx=fx$.

The transformations $fM\subset N$ and $gM\subset N$ have a *coincidence* (Lefschetz [2]) if there is an $x\in M$ with $fx=gx$. As in (C) we may show that

(D) *A monotone transformation $fM=T$ of a continuum onto a tree admits a coincidence with any continuous transformation $gM\subset T$.*

Remarks. The result (A) is usually called the Scherrer fixed-point theorem when q is single-valued and T is an acyclic continuous curve. For a list of papers concerning it see Hopf [1]. Corollary (C) will be found in [3]. The result (A) was found while constructing a proof of (D). Finally (A) is analogous to a result of S. Kakutani [4] who has shown that if S is an n-simplex and to each $s\in S$ we assign continuously a closed convex set qs then there is an $s_0\in qs_0$.

Bibliography

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5. E. Čech, Fundamenta Mathematicae, vol. 25 (1932), p. 149.

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ON THE DEFINITION OF CONTACT TRANSFORMATIONS

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If z is a function of $x_1, \cdots, x_n$ and $p_\nu = \partial z/\partial x_\nu$, $\nu = 1, \cdots, n$, a *contact transformation* in the space of $z, x_1, \cdots, x_n$, is defined by a set of $n+1$ equations

$$\text{(a)} \qquad Z = Z(z, x_\mu, p_\mu), \qquad X_\nu = X_\nu(z, x_\mu, p_\mu), \qquad \nu = 1, \cdots, n,$$

such that *firstly* in calculating the n derivatives

$$P_\nu = \frac{\partial Z}{\partial X_\nu}, \qquad \nu = 1, \cdots, n,$$

the expressions for the P_ν are given by a set of n equations

$$\text{(b)} \qquad P_\nu = P_\nu(z, x_\mu, p_\mu), \qquad \nu = 1, \cdots, n,$$

in which the derivatives of the p_μ *fall out*; and *secondly* the equations (a) and (b) can be resolved with respect to z, x_μ, p_μ:

$$\text{(A)} \qquad z = z(Z, X_\mu, P_\mu), \qquad x_\nu = x_\nu(Z, X_\mu, P_\mu), \qquad \nu = 1, \cdots, n,$$

$$\text{(B)} \qquad p_\nu = p_\nu(Z, X_\mu, P_\mu), \qquad \nu = 1, \cdots, n.$$

These two postulates are equivalent with the hypothesis that the $2n+1$ equations (a), (b) form a transformation between the two spaces of the sets of $2n+1$ independent variables (z, x_ν, p_ν), (Z, X_ν, P_ν) satisfying the Pfaffian condition

$$dZ - \sum_{\nu=1}^{n} P_\nu dX_\nu = \rho\left(dz - \sum_{\nu=1}^{n} p_\nu dx_\nu\right), \qquad \rho \neq 0.$$