A CHARACTERIZATION OF THE DISC

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In this paper the disc is characterized as the only connected, simply connected domain $B$ with the following property, $C_3$: $B$ will cover (by an isometry) any subset $P$ of the plane provided $B$ will cover each 3 points of $P$.

The disc has property $C_3$. For a plane set $P$ can be covered by a $p$-disc if and only if the members of the family $F$ of $p$-discs with centers in $P$ have a common point. If now each three points of $P$ are on a $p$-disc then each three discs of $F$ intersect and by a theorem on convex bodies due to E. Helly there is a point common to all the discs of $F$.

**Lemma.** A bounded, closed subset of the plane contains a largest circle.

The proof is accomplished by selecting a sequence of circles from the set whose centers converge to a point and whose radii converge to the least upper bound of the radii of circles in the set and (using the fact that the set is closed) showing that the limiting circle belongs to the set.

**Theorem.** The disc is the only connected, simply connected domain with property $C_3$.

**Proof.** We assume $B$ to be the given domain, and show that $B$

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1 Part of a Ph.D. dissertation at University of Missouri, under L. M. Blumenthal, 1940.
2 The disc of radius $p$ and center $p$ is the set of points $x$ of $E_2$ such that $px \leq p$.
3 Closure of a bounded open subset of $E_2$.
4 Theorem. If each $n+1$ sets of a family of bounded, closed, convex subsets of $E_n$ intersect, there is a point common to all the sets. Jahrbuch der Deutschen Mathematiker-Vereinigung, vol. 32 (1932), pp. 175–176.

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must be a disc. By the preceding lemma, $B$ contains a largest circle (center $o$ and radius $r$), and since $B$ is simply connected, the whole disc $D$ of the circle is contained in $B$. If we suppose $B$ is not a disc, there is a point $p$ of $B$ outside $D$, and, since $B$ is connected, a point $v'$ of $B$ such that $r < ov' \leq \max (op, 3^{1/2}r)$. (The inequality on the right insures that $\theta \geq 30^\circ$ in the figure.) Introduce coordinates in the plane with $o$ as origin and the ray $ov'$ as positive $X$-axis. Then $D$ is defined by the inequality

$$D: \quad x^2 + y^2 \leq r^2,$$

and the circle $K$ of the three points $(0, \pm r), v'$ has the equation

$$K: \quad \left(x - \left(\frac{v^2 - r^2}{2v}\right)\right)^2 + y^2 = \left(\frac{v^2 + r^2}{2v}\right)^2$$

where $v$ is the abscissa of the point $v'$.

The circle $K$ has the following properties:

(a) The radius of $K$ is greater than $r$. For, since $v > r$,

$$\left(\frac{v^2 + r^2}{2v}\right) > \left(\frac{v^2 - r^2}{2v}\right)^2 + r^2 > r^2.$$

(b) The arc of $K$ to the left of the $Y$-axis is of length $\geq 120^\circ$. For the arc is $2(2\theta) = 4\theta \geq 120^\circ$ by choice of $v'$.

(c) The arc of $K$ to the left of the $Y$-axis is in $D$. From (2)

$$x^2 + y^2 = r^2 + x\left(\frac{v^2 - r^2}{v}\right)$$

so that for $x \leq 0$ the inequality (1) is satisfied by points of $K$.

(d) The maximum distance of a point of $K$ from the origin is $v$. This follows easily from (3).

If, now, any three points $p_1, p_2, p_3$ of $K$ are selected, at least two of them, say $p_1, p_2$, will lie on an arc of length $\leq 120^\circ$. By property (b), $K$ can then be rotated about its center so that this arc will fall to the left of the $Y$-axis, that is, $p_1, p_2 \in D$ by property (c). Since $op_3 \leq v$ by property (d), there is, since $B$ is connected, a point $p'_3$ of $B$ at distance $op_3$ from $o$. Rotate $K$ about the origin so that $p_3$ goes into $p'_3$. Since $p_1, p_2$ will remain in $D$, the three points $p_1, p_2, p_3$ are seen to be congruent to three points of $B$. But $p_1, p_2, p_3$ are any three points of $K$, so that $B$ must, by our assumption, cover $K$. By property (a) this is impossible, and hence no point $p$ of $B$ can lie outside $D$, that is, $B$ is the disc $D$.

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