We now return to the *proof of the theorem*:

For any F in Φ there is an open set $U(F)$ such that $F\subset U(F)$, and $U(F)$ does not cut X between x_1 and x_2. Since Φ is 0-dimensional, there is an open set $V(F)$ such that $F\subset V(F)\subset U(F)$ and $[\overline{V(F)}-V(F)]\sum F=0$. By the Lindelöf covering theorem, there is a sequence $F_1, F_2, \cdots$ of elements of Φ such that $\sum F\subset\sum_{i=1}^{\infty}V(F_i)$. Now let

$$A_1 = V(F_1), \qquad A_2 = V(F_2) - \overline{V(F_1)},$$
$$A_k = V(F_k) - \overline{[V(F_1) + \cdots + V(F_{k-1})]}, \cdots .$$

The sets $A_1, A_2, \cdots, A_k, \cdots$ are open and disjoint, and no one of them cuts X between x_1 and x_2. But, as is easily shown, $\sum F\subset\sum_{k=1}^{\infty}A_k$. Hence in view of Lemma 2, $\sum F$ does not cut X between x_1 and x_2.

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SUMS OF FOURTH POWERS OF GAUSSIAN INTEGERS

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It is the purpose of this note to give necessary and sufficient conditions for the expressibility of a Gaussian integer as a sum of fourth powers of Gaussian integers; and then to determine an upper bound to the number of fourth powers necessary when the conditions are satisfied. Our results are as follows:

THEOREM. *A Gaussian integer is expressible as a sum of fourth powers of Gaussian integers if and only if its imaginary coordinate is divisible by* 24. *Every integer $a+24bi$, where a and b are rational integers, is expressible as a sum of* 18 *or fewer fourth powers.*

First we prove that the condition is necessary. We note that[1]

$$(1) \qquad (x + yi)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2).$$

It is obvious that $xy(x^2-y^2)$ is divisible by 2 and by 3. Hence any fourth power has an imaginary coordinate divisible by 24, and any sum of fourth powers has the same property.

The converse of this is included in the second statement in the theorem, which we now proceed to prove. The author[2] has shown that a Gaussian integer $a+2bi$ is expressible as a sum of two squares

[1] Latin letters will represent rational integers throughout this paper.

[2] *Integers of quadratic fields as sums of squares*, Transactions of this Society, vol. 48 (1940), p. 410, Theorem 2.

if and only if not both $a/2$ and b are odd rational integers. We use this result in the following two lemmas.

LEMMA 1. *The Gaussian integer* $6(c+di)^2$, *where* c *and* d *are odd and even, respectively, is expressible as a sum of six fourth powers.*

By the theorem stated above, we can write $c+di=\alpha^2+\beta^2$, where α and β are Gaussian integers. Also we employ the identity

$$6(\alpha^2+\beta^2)^2 = 2(\alpha+\beta)^4 + 2(\alpha-\beta)^4 + (\alpha+\beta i)^4 + (\alpha-\beta i)^4,$$

which completes the proof.

LEMMA 2. *A Gaussian integer of the form* $48h+6k+24mi$, *where* k *equals* 2 *or* 6 *according as* m *is even or odd, is expressible as a sum of twelve fourth powers.*

As above, we can write

$$8h + k + 4mi = (c+di)^2 + (e+fi)^2, \tag{2}$$

which implies

$$8h + k = c^2 + e^2 - d^2 - f^2, \qquad 2m = cd + ef. \tag{3}$$

Since $8h+k\equiv 2 \pmod 4$, the first of equations (3) shows that either c and e are odd and d and f are even, or vice versa. We show that the latter cannot be the case.

First let m be even, so that $k=2$. Then equations (3) imply the congruences

$$c^2 + e^2 - d^2 - f^2 \equiv 2 \pmod 8, \qquad cd + ef \equiv 0 \pmod 4. \tag{4}$$

Since the square of an odd integer is congruent to 1 modulo 8, the assumption that d and f are odd leads to the congruence $c^2+e^2\equiv 4 \pmod 8$. Hence the integers c and e are even, but are incongruent modulo 4. But these conditions on c and e are incompatible with the second of the congruences (4).

On the other hand, if m is odd, $k=6$, and corresponding to (4) we have the congruences

$$c^2 + e^2 - d^2 - f^2 \equiv 6 \pmod 8, \qquad cd + ef \equiv 2 \pmod 4. \tag{5}$$

Assuming again that d and f are odd, we obtain $c^2+e^2\equiv 0 \pmod 8$. Hence c and e are even, and, moreover, are congruent modulo 4. These conditions are such that the second of congruences (5) has no solutions.

Thus we have proven that d and f are even, and c and e are odd.

Multiplying equation (2) by 6, we see that Lemma 1 is applicable to each term on the right side, and we have the desired result.

To complete the proof of the theorem, we prove this lemma:

LEMMA 3. *Any integer* $a+24bi$ *is expressible as a sum of six fourth powers and an integer of the type described in Lemma* 2.

We identify b with m, and show that a is congruent to six real fourth powers modulo $48h+6k$. This we do by exhibiting the numbers $0, 1, \cdots, 47$ as sums of fourth powers modulo 48. Most of these can be handled by the use of: $1^4=1$, $2^4=16$, $(1+i)^4=-4$. Except for the values 7, 22, 23, 27, 37, 38, 39, 42, 43, and 47, all integers from 1 to 47 can be expressed as sums of 1, 16, and -4, not more than six summands being used in each case. For example, we have

$$11 = 16 - 4 - 4 + 1 + 1 + 1, \quad 46 = 16 + 16 + 16 - 4 + 1 + 1.$$

Turning now to the exceptional cases, we make use of $81=3^4$, and introduce congruences modulo 48. We can write

$$\begin{aligned} 7 &\equiv 151 = 81 + 81 - 4 - 4 - 4 + 1, \\ 22 &\equiv 70 = 81 - 4 - 4 - 4 + 1, \\ 27 &\equiv 75 = 81 - 4 - 4 + 1 + 1, \\ 37 &\equiv -11 = -4 - 4 - 4 + 1, \\ 42 &\equiv -6 = -4 - 4 + 1 + 1, \\ 47 &\equiv -1 = -4 + 1 + 1 + 1, \end{aligned}$$

these being congruences modulo 48. Since the integer 37 is represented here as a sum of four fourth powers modulo 48, the integers 38 and 39 are similarly sums of five and six fourth powers, respectively. Also the integers 23 and 43 can be compared with 22 and 42 above.

Although the theorem has been proved completely, we now show that Lemma 3 cannot be improved, that is, that six fourth powers are necessary in at least one case. Consider the situation when $a=19+48A$, $b=2B$. In this case we can show that it is not possible to obtain rational integers h and m, and Gaussian integers $\alpha_1, \cdots, \alpha_5$ to satisfy

$$a + 24bi = \sum_{i=1}^{5} \alpha_i^4 + 48h + 6k + 24mi, \tag{6}$$

where, of course, k is 2 or 6 according as m is even or odd.

LEMMA 4. *If the imaginary part of the fourth power of a Gaussian integer is congruent to* 0 *modulo* 48, *the real part is congruent to* 0, 1, *or* 12 *modulo* 16. *If the imaginary part is congruent to* 24 *modulo* 48, *the real part is congruent to* 9 *modulo* 16.

From equation (1) it follows that

$$R(x+yi)^4 = (x^2 - y^2)^2 - 4x^2y^2, \quad I(x+yi)^4 = 4xy(x^2 - y^2).$$

Since the square of a rational integer is congruent to 0, 1, 4 or 9 modulo 16, it is easily verified that $R(x+yi)^4$ is congruent to 0, 1, 9, or 12 modulo 16. Also note that $R(x+yi)^4$ is congruent to 9 modulo 16 if and only if x is odd and y is congruent to 2 modulo 4, or vice versa; and when x and y satisfy these conditions, $I(x+yi)^4$ is congruent to 24 modulo 48. In all other cases either xy or x^2-y^2 is divisible by 4, and hence $I(x+yi)^4$ is congruent to 0 modulo 48.

We now use Lemma 4 to prove equation (6) impossible. First let m be even, so that $k=2$. Since b is even the imaginary parts of (6) can be equal only if $\sum\alpha_i^4$ has an imaginary coordinate congruent to 0 modulo 48. This implies that an even number of the terms $I(\alpha_i^4)$ are congruent to 24 modulo 48. In turn, Lemma 4 states that an even number of the terms $R(\alpha_i^4)$ are congruent to 9 modulo 16. The real parts of equation (6) can be written as a congruence:

$$19 \equiv \sum_{i=1}^{5} R(\alpha_i^4) + 12 \pmod{16}.$$

Simple verification shows that this congruence cannot be satisfied by assigning the values 0, 1, 9, or 12 to the terms $R(\alpha_i^4)$, the value 9 being used an even number of times.

Second let m be odd, so that $k=6$. Corresponding to the above congruence we have

$$19 \equiv \sum_{i=1}^{5} R(\alpha_i^4) + 36 \pmod{16}.$$

In this case, however, Lemma 4 requires that we have an odd number of the terms $R(\alpha_i^4)$ congruent to 9 modulo 16. Again it can be verified that the congruence has no solutions satisfying this condition.

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