EVERYWHERE DENSE SUBGROUPS OF LIE GROUPS

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A recent note by Montgomery and Zippin[1] leads one to speculate concerning the nature of everywhere dense proper subgroups of continuous groups. Such subgroups can easily be constructed. Suppose for example that G is a non-countable continuous group which admits a countable subset G_0 filling it densely. The group generated by G_0 is everywhere dense in G but is not identical with G. In the case of Lie groups, it is easy to see that an abelian G admits non-countable subgroups of the sort in question; whether or not a non-abelian G does so, appears to be a more difficult question. We shall, however, show that if G is simple, proper subgroups of G cannot, so to speak, fill G *too* densely.

Let G be a simple[2] Lie group of dimension r with $r>1$, and let U be a canonical nucleus of G—that is, a nucleus which can be covered by an analytic canonical coordinate system. An arbitrary point x of U is contained in the central of at least one closed proper Lie subgroup of G with non-discrete central. In fact, through x there passes a one-parameter subgroup γ; the closure of γ is an abelian Lie subgroup and this subgroup is proper since G is simple and $r>1$.

THEOREM. *Let G be a simple Lie group of dimension r greater than one and let $\mathfrak{g}$ be a proper subgroup filling G densely. There exists at least one proper closed Lie subgroup H of G such that those left- (right-) cosets of H which fail to meet $\mathfrak{g}$ fill G densely. For H one may take any closed proper Lie subgroup of G whose central is non-discrete and contains an arbitrarily chosen point p in $\mathfrak{g}\cap U$, U being any given canonical nucleus of G.*

PROOF. Let U, p, H be chosen and let us consider only the left-cosets of H. It will be sufficient to prove that there exists at least one coset, say aH, which fails to meet $\mathfrak{g}$. For, the cosets obtained by multiplying aH on the left by arbitrary elements of $\mathfrak{g}$ fail to meet $\mathfrak{g}$ and fill G densely.

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[1] Deane Montgomery and Leo Zippin, *A theorem on the rotation group of the* 2-*sphere*, this Bulletin, vol. 46 (1940), pp. 520–521. Our theorem may be regarded as a generalization of the theorem of Montgomery and Zippin and the proofs of the two theorems may be regarded as being the same in principle.

[2] We use *simple* here in the sense of having a simple Lie algebra. A simple group need not be connected.

Let us assume the contrary, namely that *every* coset of H meets $\mathfrak{g}$. Let H^* be the totality of cosets of H and let the elements of H^* be denoted by $e^*=H$, $a^*=aH$, $\cdots$. Let σ be the mapping $x\rightarrow x^*$ $(x^*=xH)$ of G into H^*. Let H^* be topologized in the usual way by taking as open in H^* every set of the form σA where A is an open subset of G. The space H^* is homogeneously locally euclidean.—Now let x^* be an element of H^* and let x be a representative of the *coset* x^*. Then xpx^{-1} (where p is defined in the theorem) is independent of x. For if y is a second representative of x^*, then $x^{-1}y\subset H$ so that $x^{-1}yp=px^{-1}y$ since p is in the central of H. Hence $xpx^{-1}=ypy^{-1}$. Thus the formula $\tau(x^*)=xpx^{-1}$ defines a mapping τ of H^* into G which, in particular, carries e^* into p. Evidentally τ is continuous. In fact it is easy to see that τ is analytic relative to an arbitrarily chosen analytic canonical coordinate system $x_1, \cdots, x_r$ covering U, and a suitably chosen coordinate system covering a neighborhood of e^*.

The mapping τ carries H^* into a subset of $\mathfrak{g}$. For, by our assumption on the cosets of H, an element y^* of H^* can be written in the form $y^*=gH$ where $g\subset\mathfrak{g}$. Hence we have $\tau(y^*)=gpg^{-1}\subset\mathfrak{g}$.—Moreover, any given neighborhood V^* of e^* contains at least one point x^* such that $\tau(x^*)\neq p$. For otherwise we have $\tau(yH)=p$ for every y in a certain nucleus V of G, that is, for every y in V and h in H we have $yhp(yh)^{-1}=p$ or $ypy^{-1}=p$. But then the one-parameter subgroup of G determined by p would be invariant, contrary to the hypothesis that G is simple.

Let W be a nucleus of G such that $W^{-1}WW\subset U$. It follows from the last two paragraphs that there exists in H^* a point z^* near e^* such that the linear segment e^*z^* is carried by τ into an analytic arc contained in $\mathfrak{g}\cap W$ and consisting of more than a single point. A suitably chosen piece of this arc, when multiplied on the left by the inverse of one of its points, furnishes an analytic 1-cell K contained in $\mathfrak{g}\cap W$ and containing e, the identity of G. Starting with K we shall construct a dimensionally increasing sequence of analytic continua, subsets of $\mathfrak{g}$. In what follows, let it be understood that all functions are real, single-valued and analytic over the domains indicated.

We may suppose that K is defined parametrically, say by $x_i=f_i(t)$ where $-1<t<1$ and $f(0)=e$. The set KK is in $\mathfrak{g}$ and is defined by equations of the form $x_i=g_i(s, t)$ where $-1<s, t<1$. Suppose that $\dim KK>\dim K$; that is, suppose $\dim KK=2$. Then being an analytic locus, KK contains points at which it is locally euclidean 2-dimensional. If b is such a point, then $b^{-1}KK$ (a subset of $\mathfrak{g}$) is locally euclidean at e. Hence $\mathfrak{g}\cap W$ contains a 2-cell K_2 defined say by $x_i=h_i(u, v)$ where $-1<u, v<1$ and $h(0, 0)=e$. We next consider the set K_2K_2 and suppose that its dimension exceeds that of K_2. On continuing in this manner, we finally obtain a k-cell E in $\mathfrak{g}\cap W$ defined say by $x_i=h_i(u_1, \cdots, u_k)$ where $-1<u_i<1$ and $h(0, \cdots, 0)=e$, and such that dim $EE=$dim $E=k$. We assert that E contains subsets E^* and F such that (1) E^* and F are k-cells; (2) $e\subset F\subset E^*$; (3) $FF\subset E^*$.

To prove this, we first note that by the theory of implicit functions, E contains a k-dimensional sub-cell E^* definable, after renaming the coordinates x_i if necessary, by equations

$$x_i = X_i(x_1, \cdots, x_k), \qquad i = k+1, \cdots, r, \tag{1}$$

where $(x_1, \cdots, x_k)$ ranges over the cube C_δ: $-\delta<x_i<\delta$, and where $X_i(0, \cdots, 0)=e_i=0$. On replacing δ by a smaller number if necessary, it is easy to see that C_δ contains a cube C_μ: $-\mu<x_i<\mu$ $(i=1, \cdots, h)$ such that if F is the k-cell defined by (1) with $(x_1, \cdots, x_k)$ restricted to the cube C_μ, and if q is an arbitrary point of F, then qF, like F, is definable by equations of the form (1):

$$x_i = X_i^q(x_1, \cdots, x_k)$$

where $(x_1, \cdots, x_k)$ ranges over a certain open subset A^q of C_δ. Now EE is the union of k-cells qE $(q\subset E)$, hence is k-dimensional at every point. Being an analytic locus, the points q at which EE is locally euclidean k-dimensional fill it densely. Consider such a point q. The k-cells F and qF intersect at q. But since both are contained in EE which is locally euclidean k-dimensional at q, they coincide identically in the neighborhood of q. Hence the functions X_i and X_i^q are identically equal over an open subset of A^q; hence, by the theory of analytic functions, they are equal over the whole of A^q. Hence $qF\subset E^*$, and this is true for a set of points q filling F densely. By continuity this relation holds for arbitrary q in F. Hence $FF\subset E^*$, proving our assertion.

It is easy to see that on replacing F by a smaller k-cell if necessary, we have also $F^{-1}\subset E^*$. In short F is a k-dimensional local Lie subgroup of G; hence it is an open subset of a k-dimensional linear subspace L of the linear space of the canonical coordinates $x_1, \cdots, x_r$. If $k<r$, there exists in W an element a such that the linear subspace L' determined by $F'=aFa^{-1}$ is different from L; otherwise the Lie subalgebra represented by L is invariant. Since $\mathfrak{g}$ is everywhere dense in G, we may assume, so far as the relation $L\neq L'$ is concerned, that $a\subset\mathfrak{g}$. Then $FF'\subset\mathfrak{g}$. Moreover, it is evident that dim $FF'>k$. We can now repeat the construction described above starting with a suitably chosen analytic cell of dimension exceeding k in FF'. We obtain finally an analytic r-cell contained in $\mathfrak{g} \cap W$. Hence $\mathfrak{g}$ contains a nucleus of G and hence $\mathfrak{g} = G$, a contradiction which proves the theorem.[3]

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[3] We have proved, incidentally, that if an everywhere dense subgroup $\mathfrak{g}$ of a simple Lie group G_r $(r > 1)$ contains an analytic arc, then $\mathfrak{g} = G$.

VECTOR SPACES OVER RINGS

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1. Introduction. Let $\mathfrak{M} = u_1K + \cdots + u_mK$ be a vector space (linear form modul [**5**, p. 111]) over a ring $K = \{0, \alpha, \beta, \cdots ; \epsilon \text{ unit element}\}$. By a *submodul* $\mathfrak{N} \leqq \mathfrak{M}$ is meant an "admissible" submodul: $\mathfrak{N}K \leqq \mathfrak{N}$. Elements $v_1, \cdots, v_n$ of a submodul $\mathfrak{N}$ form a *basis* for $\mathfrak{N}$ (notation: $\mathfrak{N} = v_1K + \cdots + v_nK$) in case $\sum v_i\alpha_i = 0$ implies $\alpha_i = 0$, $i = 1, \cdots, n$, and if every element of $\mathfrak{N}$ is expressible in the form $\sum v_i\alpha_i$, $\alpha_i \in K$. The equivalent formulations of the ascending chain condition for submoduls of a vector space, and for right ideals of a ring will be used without further comment [**5**, §§80, 97].

2. Basis number, linear transformations. We remark that the following holds.

(A) *The ascending chain condition is satisfied by the submoduls of a vector space $\mathfrak{M}$ over K if and only if it is satisfied by the right ideals of K.*

An infinite chain of right ideals $\mathfrak{r}_1 < \mathfrak{r}_2 < \cdots$ in K yields an infinite chain of submoduls $u_1\mathfrak{r}_1 < u_1\mathfrak{r}_2 < \cdots$ in $\mathfrak{M}$. The other implication is proved in [**5**, p. 87].

[By using a lemma due to N. Jacobson (*Theory of Rings*, in publication) Theorem (A) and the corresponding theorem for descending chain condition are easily proved in a unified manner.]

Linear transformations of $\mathfrak{M}$ on $\mathfrak{M}$ are given by $u_j \rightarrow u_j' = \sum u_i\alpha_{ij}$. Write $(u_1', \cdots, u_m') = (u_1, \cdots, u_m)A$, $A = (\alpha_{ij})$. Under $u_j \rightarrow u_j'$, let $\mathfrak{M}_0 \rightarrow 0$. Thus $\mathfrak{M}/\mathfrak{M}_0 \cong \mathfrak{M}A \leqq \mathfrak{M}$. Clearly $\mathfrak{M}_0 = 0$ if and only if $Av = 0$ implies $v = 0$, v an $m \times 1$ matrix over K, and $\mathfrak{M}A = \mathfrak{M}$ if and only if there exists an $m \times m$ matrix R with $AR = I$, the identity matrix.

Possibilities (i) $\mathfrak{M}_0 = 0$ and $\mathfrak{M}A = \mathfrak{M}$; (ii) $\mathfrak{M}_0 > 0$ and $\mathfrak{M}A < \mathfrak{M}$; (iii) $\mathfrak{M}_0 = 0$ and $\mathfrak{M}A < \mathfrak{M}$ are familiar. The possibility of (iv) $\mathfrak{M}_0 > 0$

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