ON 3-DIMENSIONAL MANIFOLDS

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Let $P$ be a 3-dimensional manifold. Let $Q$ be a 2-dimensional manifold imbedded in $P$. Moreover, let $P$ and $Q$ admit of a permissible simplicial division $K$, that is, a simplicial division of $P$ such that some subcomplex of $K$, say $L$, is a simplicial division of $Q$. Let $K_i$ and $L_i$ denote the $i$th normal subdivisions of $K$ and $L$, respectively. We define the neighborhood $N_i$ of $L_i$ to be the simplicial complex consisting of the simplexes of $K_i$ that have at least one vertex in $L_i$, together with the sides of all such simplexes. By the boundary $B_i$ of $N_i$ we mean the simplicial complex consisting of the simplexes of $N_i$ that have no vertex in $L_i$. Our purpose is to prove the following theorem.

**Theorem.** The boundary $B_2$ is a two-fold but not necessarily connected covering of $Q$, and change of permissible division $K$ replaces $B_2$ by a homeomorph of itself.

**Proof.** The neighborhood $N_1$ is the sum of a set of 3-dimensional simplexes. Some of these 3-simplexes, say $a_1, a_2, \ldots$, have exactly one vertex in $L_1$, others, say $b_1, b_2, \ldots$, have exactly two vertices in $L_1$, while the remaining, say $c_1, c_2, \ldots$, have three vertices in $L_1$. Since $K_1$ is a normal subdivision of $K$, the intersection of $L_1$ and $b_i$ or $c_i$ is a 1-simplex or 2-simplex, respectively. Let $\alpha_i, \beta_i,$ and $\gamma_i$ be the intersections of $B_2$ and $a_i, b_i,$ and $c_i,$ respectively. We shall regard $\alpha_i$ and $\gamma_i$ as triangles with vertices on the 1-simplexes of $a_i$ and $c_i$. Also we shall regard $\beta_i$ as a square with vertices on the 1-simplexes of $b_i$.

Any 2-simplex of $L_1$, say $ABC$, is incident to exactly two of the $c_i$. Let $c_1 = ABCM$. There is a unique 3-simplex of $N_1$, say $\sigma$, that is incident to $ABM$ and different from $c_1$. This $\sigma$ is either a $c_2$, say $c_2$, or a $b_i$, say $b_2$. If $\sigma = c_2$, then the triangles $\gamma_1$ and $\gamma_2$ have a common side. Suppose that $\sigma = b_2 = ABMN$. The 2-simplex $ABN$ is incident to a unique 3-simplex of $N_1$, say $\tau$, with $\tau \neq ABMN$. This $\tau$ is either $c_3$ or $b_3$. If $\tau = b_3$, there is a $c_4$, or $b_4$. Finally we must find a $c_p = ABDS$, $D$ in $L_1$, $S$ in $B_1$. We now consider $\beta_2, \beta_3, \ldots,$ and $\beta_{p-1}$. The sum of these squares is topologically equivalent to a square. One side of the square is coincident with a side of $\gamma_1$ and the opposite side coincident with a side of $\gamma_p$.

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Our terminology is that of Seifert-Threlfall, *Lehrbuch der Topologie*. Manifolds are finite, while simplexes and cells are closed point sets.
Since $K_1$ is a manifold, we can repeat the construction and associate with $ABC$ and $ABD$ a second pair of triangles in $B_2$ that are either incident along a common side or incident to opposite sides of a square. But there is not a third such configuration associated with $ABC$ and $ABD$. We repeat the construction for all pairs of adjacent 2-simplexes of $L_1$. Then to each 2-simplex of $L_1$ there correspond two triangles in $B_2$. Moreover, if two 2-simplexes of $L_1$ are incident along a side, the four corresponding triangles can be paired so that the two triangles of each pair either have a common side or are incident to opposite sides of a square.

Since $P$ and $Q$ are 3- and 2-manifolds, respectively, we can say that $Q$ is two-sided in $P$ in the neighborhood of any point of $Q$. Moreover, the two $\gamma$'s of $B_2$ that correspond to a 2-simplex of $L_1$ lie on opposite sides of $Q$ (in the neighborhood of this 2-simplex).

Consider a vertex $X$ of $L_1$ and the 2-simplexes $\Delta_i$ of $L_1$ that have $X$ as a vertex. On one side of $Q$ (in the neighborhood of $X$) there corresponds to each $\Delta_i$ a unique $\gamma_i$, and the $\gamma$'s have the same incidences as the corresponding $\Delta$'s (we say that two $\gamma$'s are incident if they are incident to opposite sides of a square). Let us denote by $R$ the points of these $\gamma$'s and the squares incident to pairs of these $\gamma$'s. Let $A$ denote the points of all $\alpha_i$'s that are in $a_i$'s incident to $X$ and on the side of $Q$ that we are considering.

We shall show that $R+A$ is a 2-cell. To do this we shall show that $R+A$ is a manifold relative to its boundary, that its boundary consists of one or more circles, and that any 1-cycle of $R+A$ bounds in $R+A$. First we observe that $B_2$ is a manifold; this fact follows from the structure of $B_2$ and the fact that $K_1$ is a manifold; the argument is elementary and we omit it. Since $R+A$ is the sum of 2-cells $\alpha$, $\beta$, and $\gamma$, the set $R+A$ is a manifold relative to its boundary.

To show that this boundary of $R+A$ consists of one or more circles we shall study the incidences among the cells of $R+A$. First, let $a_i$ have $X$ as a vertex. If a 2-dimensional side of $a_i$ is not in $B_1$, this side must be a side of an $a_j$ or $b_j$. Furthermore, this $a_j$ or $b_j$ has $X$ as a vertex. Hence, any side of an $\alpha_i$ is also a side of an $\alpha_j$ or $\beta_j$ of $R+A$. Next, let $c_i$ have vertices $XABM$, $M$ in $B_1$. The sides of $\gamma_i$ that are in $XAM$ and $XBM$ are sides of $\gamma_j$'s or $\beta_j$'s of $R+A$. But the side of $\gamma_i$ in $ABM$ is not incident to any other 2-cell of $R+A$. This side is part of the boundary of $R+A$. Finally, let $b_i$ have vertices $XAMN$, $A$ in $L_1$. The sides of $\beta_i$ in $XAM$ and $XAN$ are incident to sides of $\beta_j$'s or $\gamma_j$'s of $R+A$; the side of $\beta_i$ in $XMN$ is incident to an $\alpha_j$ or $\beta_j$ of $R+A$; but the side of $\beta_i$ in $AMN$ is not incident to any other 2-cell of $R+A$. This side is part of the boundary of $R+A$. Examination of
the segments of the boundary of $R+A$ shows that they fit together to form one or more circles.

We next show that if $C$ is a 1-dimensional cycle of $R+A$, then $C$ bounds in $R+A$. We shall find it convenient to replace $A$ by a new set that will never be empty. We define $A'$ to be $A$ together with all vertices of $\gamma$'s of $R$ that are not in the boundary of $R+A$ and all sides of squares of $R$ that are not sides of $\gamma$'s of $R$ and not in the boundary of $R+A$. If $A$ is not empty, the set $A'$ is the same as $A$. But in any case $A'$ is not empty, and $R+A'$ is the same set as $R+A$. The set $(R+A')-\bar{A}'$ is homeomorphic to a 2-cell with an inner point removed because $(R+A')-\bar{A}'$ can be obtained from the configuration of the 2-simplexes of $L_1$ that have $X$ as a vertex by removing $X$ and replacing some of the 1-simplexes by squares (open along one side). Hence, the cycle $C$ is homologous in $R+A'$ to a cycle on $A'$, and we assume that $C$ is on $A'$. The set $A'$ is part of $b$, the boundary of the combinatorial neighborhood of $X$ in $K_2$. Since $K_2$ is a manifold, the set $b$ is a 2-sphere. Assume that $C$ does not bound in $A'$. Then $C$ must surround a 2-simplex of $b$ that is not in $A'$. We easily find a 2-simplex of $R+A'$ that is not incident along one of its sides to another 2-simplex of the manifold $B_2$. This contradiction proves that $C$ bounds, and the proof that $R+A$ is a 2-cell is complete.

Now we draw some lines on $R+A$. If two $\gamma$'s have a common side, we draw a line coincident with this common side. If two $\gamma$'s are incident to a square, we draw a line across the square half way between the $\gamma$'s. All these lines are continued so that they meet at a point of $A$. These lines give a subdivision of $R+A$ that is combinatorially equivalent to the combinatorial neighborhood of $X$ in $L_1$. The lines can be drawn for all $R+A$ of $B_2$ and we get a subdivision of $B_2$ that is combinatorially equivalent to a two-fold but not necessarily connected covering of $L_1$.

A triangle of the covering is associated with a 2-simplex of $L_1$ and a side of $Q$ (in the neighborhood of this simplex). Hence, a homeomorphism is determined between this covering and any covering obtained by changing the permissible division $K$.

The theorem is not true with $B_1$ rather than $B_2$. For example, let $Q$ be the boundary of a 3-simplex of $K$. Then $B_1$ is a sphere and a point.

We can prove the following theorem in the same way but with much less effort.

**Theorem.** The above theorem is true if $P$ and $Q$ are replaced by 2- and 1-dimensional manifolds.

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