be the orthonormal trigonometric sums on $D_1$ for weight $\rho(\theta)\sigma(\theta)$; if the $u$'s and $v$'s are uniformly bounded on a point set $D_2$ contained in $D_1$, the same is true of the $U$'s and $V$'s.

For this case the proof admits a materially simpler formulation than when geometric configurations are contemplated having the degree of generality previously considered. The details relating to the loci $C', C'', K, K', K''$, can be dispensed with for the most part; with $\theta$ replacing the pair of coordinates $(x, y)$, and $\phi$ replacing the pair $(u, v)$, it is sufficient, for any particular value of $\theta$, to consider separately the intervals $(\theta - \pi/2, \theta + \pi/2)$ and $(\theta + \pi/2, \theta + 3\pi/2)$, and in the integral corresponding to the right-hand member of (5) to represent $K_{n-1}(\theta, \phi)$ in the former interval by an expression with denominator $\sin (\theta - \phi)$, and in the latter interval by an alternative expression with $1 - \cos (\theta - \phi)$ in the denominator.\(^{10}\)

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\(^{10}\) [B, pp. 808–809.]

APPROXIMATION OF CONTINUOUS FUNCTIONS BY MEANS OF LACUNARY POLYNOMIALS

BERNARD DIMSDALE

1. Introduction. All rational integral polynomials are linear combinations of members of the complete set of powers whose exponents are the non-negative integers. If certain members of this set are deleted, the linear combinations formed from the resulting set are, in the strict sense of the term, "lacunary polynomials." In a large part of this paper, however, methods of reasoning designed for the treatment of such polynomials are applicable to combinations from much more general sets of powers whose exponents are non-negative but not in general integral. The term "polynomial in $x^\mu$ of degree $\mu_n$" will be applied to combinations from the set $1, x^{\mu_1}, x^{\mu_2}, \ldots$ where $\mu_1, \mu_2, \ldots$ form an arbitrarily preassigned set of real numbers such that $0 < \mu_1 < \mu_2 < \ldots$, and $\mu_n$ is the largest exponent.

This paper started out as an investigation of lacunary orthogonal polynomials, and although this aspect of it became subordinate to the

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problem of approximation, the relation of the latter to the theory of convergence of series of the orthogonal polynomials is pointed out at the end.

2. Preliminary results. It has been shown\(^1\) that for all positive integers \(m\) the trigonometric sum

\[
I_m(t) = h_m \int_{-\pi/2}^{\pi/2} g(t + 2u)F_m(u)du
\]

with

\[
F_m(u) = \left[ \frac{\sin mu}{m \sin \frac{u}{m}} \right]^4, \quad \frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} F_m(u)du,
\]

of order \(2m - 2\), satisfies

\[
|I_m(t) - g(t)| \leq D\lambda/m, \text{ for all } t,
\]

provided that \(g(t)\) has period \(2\pi\) and satisfies a Lipschitz condition, constant \(\lambda\), for all real \(t\), where \(D\) is an absolute constant; and, furthermore, that if \(g(t)\) is an even function, \(I_m(t)\) will be a cosine sum, \(I_m(t) = \alpha_0^{(m)} + \alpha_1^{(m)} \cos t + \cdots + \alpha_{2m-2}^{(m)} \cos (2m-2)t\).

Now

\[
\alpha_k^{(m)} = \frac{1}{\pi} \int_0^{2\pi} I_m(t) \cos kt dt,
\]

and when \(I_m(t)\) is replaced by its integral form, order of integration reversed, and the substitution \(t + 2u = y\) made,

\[
\alpha_k^{(m)} = \frac{h_m}{\pi} \int_{-\pi/2}^{\pi/2} F_m(u) \left[ \int_0^{2\pi} \cos (y - 2u)g(y)dy \right]du,
\]

since \(g(t)\) has period \(2\pi\). Since\(^2\)

\[
\left| \frac{1}{\pi} \int_0^{2\pi} g(t) \cos kt dt \right| \leq \frac{\pi \lambda}{k}
\]

it follows on expanding \(\cos k(y - 2u)\) that\(^3\)

\[
|\alpha_k^{(m)}| \leq \frac{h_m}{\pi} \int_{-\pi/2}^{\pi/2} F_m(u) \frac{\pi^2 \lambda}{k} du + \frac{h_m}{\pi} \int_{-\pi/2}^{\pi/2} F_m(u) \frac{\pi^2 \lambda}{k} du,
\]

\[
|\alpha_k^{(m)}| \leq \frac{2\pi \lambda}{k}, \quad k = 1, 2, \ldots, m - 1; m = 2, 3, \ldots.
\]

---


\(^2\) See, for example, Titchmarsh, *The Theory of Functions*, p. 426.

\(^3\) The following shorter proof has been given \(|\alpha_k^{(m)}| = \left| (1/\pi) \int_0^{2\pi} [I_m(t) - g(t) + g(t)] \cos kt \right| \leq 2D\lambda/m + n\lambda/k \leq \lambda(2D + \pi)/k\). The different value of the constant here would change subsequent computations slightly.
If \( g(x) \) satisfies a Lipschitz condition, constant \( \lambda \), on the closed interval \([-1, 1]\), then for each positive integer \( m \) there exists a polynomial \( P_m(y) \) in \( 1, y, y^2, \cdots \), of degree \( 2m - 2 \), such that on \([-1, 1]\)

\[
\left| g(y) - P_m(y) \right| \leq \frac{D\lambda}{m}.
\]

Here\(^6\) \( P_m(y) = I_m(t) \), where \( I_m(t) \) is defined for \( g(\cos t) \) and \( y = \cos t \). Let \( P_m(y) = a_0^{(m)} + a_1^{(m)} y + \cdots + a_{2m-2}^{(m)} y^{2m-2} \). The following inequalities result directly from the fact that \( P_m(y) = I_m(t) \);

\[
\begin{align*}
\left| a_k^{(m)} \right| & \leq \frac{2^{2k-1} \pi \lambda}{2k} C_{2k}(m + k - 1), \\
\left| a_{2k-1}^{(m)} \right| & \leq \frac{2^{2k-1} \pi \lambda}{2k - 1} C_{2k-1}(m + k - 2),
\end{align*}
\]

where

\[
C_r(s) = \frac{s!}{(s - r)! r!}, \quad k = 1, 2, \cdots, m - 1; m = 2, 3, \cdots.
\]

Let \( y = x^H, g(x^H) = f(x), H > 0 \). Then\(^8\) \( \left| f(x) - f(x_1) \right| \leq \lambda \left| x^H - x_1^H \right| \) on \([0, 1]\), implies that for every positive integer \( m \) there exists a polynomial

\[
P_{mH}(x) = a_0^{(m)} + a_1^{(m)} x^H + \cdots + a_{2m-2}^{(m)} x^{(2m-2)H}
\]

such that on \([0, 1]\),

\[
\left| f(x) - P_{mH}(x) \right| \leq \frac{D\lambda}{m}.
\]

It will be observed that in this transformation the coefficients \( a_k^{(m)} \) remain unchanged.

It has been shown by Szász\(^7\) that for a sequence \( \gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_n \), where \( 0 < \gamma_0, 0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n \) and \( \gamma_0 \neq \gamma_i, i = 1, 2, \cdots, n \), there exists a set of constants \( u_1, u_2, \cdots, u_n \) such that\(^7\)

---

\(^4\) Hereafter the bracket notation for an interval implies closure at both ends.


\(^6\) Hereafter this condition will be called a "Lipschitz condition \( H \)." It has been pointed out that this condition implies that \( f(x) \) satisfies a Lipschitz condition of order \( \alpha \), \( \alpha = H \) if \( H \leq 1 \), \( \alpha = 1 \) if \( H \geq 1 \). However, the converse is not true, since \( x = 0 \) belongs to the interval in question. Hence this condition is more restrictive than a Lipschitz condition of order \( \alpha \). On any closed interval not including \( 0 \) the two conditions are, however, equivalent.

\[
\int_0^1 \left[ x^{\gamma_1-1/2} + u_1 x^{\gamma_2-1/2} + \cdots + u_n x^{\gamma_n-1/2} \right]^2 dx \leq \frac{1}{2\gamma_0} \prod_{h=1}^{n} \left[ \gamma_h - \gamma_0 \right]^2.
\]

Let
\[
A(x) = \frac{x^{\gamma_0+1/2}}{\gamma_0 + 1/2} + \frac{u_1 x^{\gamma_1+1/2}}{\gamma_1 + 1/2} + \cdots + \frac{u_n x^{\gamma_n+1/2}}{\gamma_n + 1/2},
\]
\[
B(x) = \frac{(\gamma_0 + 1/2)A(x)}{x^{1/2}} = x^{\gamma_0} + c_1 x^{\gamma_1} + \cdots + c_n x^{\gamma_n},
\]
\[
C(x) = x^{\gamma_0-1/2} + u_1 x^{\gamma_1-1/2} + \cdots + u_n x^{\gamma_n-1/2}.
\]
Then
\[
A(x) = \int_0^x C(x) dx,
\]
and
\[
[A(x)]^2 = \left[ \int_0^x C(x) dx \right]^2 \leq \int_0^x C(x)^2 dx \leq x \cdot \int_0^1 C(x)^2 dx \leq \frac{x}{2\gamma_0} \prod_{h=1}^{n} \left[ \gamma_h - \gamma_0 \right]^2.
\]

Therefore on \([0, 1]\]
\[
|A(x)| \leq \frac{x^{1/2}}{2\gamma_0} \prod_{h=1}^{n} \left| \frac{\gamma_h - \gamma_0}{\gamma_h + \gamma_0} \right|,
\]
\[
|B(x)| \leq \frac{2\gamma_0 + 1}{2(2\gamma_0)^{1/2}} \prod_{h=1}^{n} \left| \frac{\gamma_h - \gamma_0}{\gamma_h + \gamma_0} \right|.
\]

If \(\gamma_0 = \gamma_i\) for some \(i\), then the \(u\)'s, and hence the \(c\)'s, may obviously be chosen so that \(|B(x)| = 0\), and the inequality still holds.

It may now be shown that if \(\gamma_0 = rH, (i-1)H < \gamma_i \leq iH\), then
\[
L_{r,n}(x) = |B(x)| \leq \frac{2rH + 1}{2(2rH)^{1/2}} \cdot \frac{1}{C_r(2r - 1) \cdot C_{2r}(n + r)}
\]
on \([0, 1]\],

for \(1 \leq r \leq n\). The proof depends upon the fact that for \(\gamma_0 \geq \gamma_h\),
\[
\frac{\gamma_0 - \gamma_h}{\gamma_0 + \gamma_h} \leq \frac{rH - (h - 1)H}{rH + (h - 1)H} = \frac{r - h + 1}{r + h - 1}.
\]

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with a similar situation for \( \gamma_0 \leq \gamma_i \). If, in particular \( \gamma_i = iH \) for some \( i \), \( \gamma_0 = iH \), then \( L_{r,n}(x) = 0 \), and the inequality is still true.

3. Degree of approximation. This theorem will now be proved.

**Theorem I.** Consider the sequence \( \mu_0, \mu_1, \mu_2, \cdots \), for which \( 0 = \mu_0 < \mu_1 < \mu_2 < \cdots \), where \( \mu_{n+1} - \mu_n \leq H \) for all \( n \), and \( \lim_{n \to \infty} \mu_n = \infty \). If \( f(x) \) satisfies a Lipschitz condition \( H \), constant \( \lambda \), on \( [0, 1] \), then for every integer \( n > 0 \) there exists \( P_{\mu_n}(x) \), a polynomial in \( x^\mu \) of degree \( \mu_n \), such that on \( [0, 1] \) \( |f(x) - P_{\mu_n}(x)| \leq K\lambda/\mu_n \), where \( K \) depends on \( H \) only.

To prove this theorem, a polynomial \( P_{\mu_n}(x) = a_0^{(\mu)} + a_1^{(\mu)} x + \cdots + a_{2m-2}^{(\mu)} x^{(2m-2)H} \) is constructed so that on \( [0, 1] \) \( |f(x) - P_{\mu_n}(x)| \leq D\lambda/m \), with bounds on the \( a \)'s as given in the last section. Unless \( rH = \gamma_k \), each \( x^{\gamma_k} \) is then approximated by a polynomial \( P_{\gamma_k}^{(\mu)}(x) = -c_0 x + \cdots - c_{4m-4} x^{4m-4} \), where the set of \( \gamma_k \)'s is a subsequence of the set of \( \mu_k \)'s, \( \gamma_0 = \mu_0 \), \( \gamma_i \) is the smallest \( \mu_k \) such that \( (i-1)H < \gamma_i \leq iH \), \( i \geq 1 \); and \( |x^{\gamma} - P_{\gamma_k}^{(\mu)}(x)| = L_{r,4m-4}(x) \). Let

\[
P_{\gamma_k}^{(\mu)}(x) = a_0^{(\mu)} + \sum_{k=1}^{2m-2} a_k^{(\mu)} P_{\gamma_k}^{(\mu)}(x).
\]

Then

\[
|f(x) - P_{\gamma_k}^{(\mu)}(x)| \leq \frac{D\lambda}{m}, \quad \text{if } m = 1,
\]

\[
\leq \frac{D\lambda}{m} + \sum_{k=1}^{m-1} P_k + \sum_{k=1}^{m-1} Q_k, \quad \text{if } m = 2, 3, \cdots,
\]

where \( P_k \) and \( Q_k \) are the respective bounds of \( |a_0^{(\mu)}| \cdot |L_{2k,4m-4}(x)| \) and \( |a_{2k-1}^{(\mu)}| \cdot |L_{2k-1,4m-4}(x)| \), as previously obtained. On calculation it is found that

\[
\frac{P_{k+1}}{P_k} = \frac{4(k+1)H + 1}{4kH + 1} \cdot \frac{k^{1/2}}{(k+1)^{1/2}} \cdot \frac{2k+1}{4m+2k-3} \cdot \frac{m+k}{2m+k-1} \cdot \frac{k}{m-k-1},
\]

\[
\frac{Q_{k+1}}{Q_k} = \frac{2(2k+1)H + 1}{2(2k-1)H + 1} \cdot \frac{(2k-1)^{1/2}}{(2k+1)^{1/2}} \cdot \frac{2k}{4m+2k-3} \cdot \frac{m+k-1}{2m+k-2} \cdot \frac{2k-1}{m-k-1},
\]

for \( k = 1, 2, \cdots, m-2; m = 3, 4, 5, \cdots \). If now each fraction is re-
placed by its maximum for the given range of \( k \) with \( m \) fixed, and then by its maximum for the given range of \( m \), it follows that, since 
\[
(8H+1)/(4H+1) \leq 2, \quad (6H+1)/(2H+1) \leq 3, \quad P_{k+1}/P_k \leq 2/9, \quad Q_{k+1}/Q_k \leq 1/3; \quad \text{hence} \quad P_k \leq (2/9)^{k-1}P_1, \quad Q_k \leq (1/3)^{k-1}Q_1.
\]
Finally
\[
|f(x) - P_{\gamma_4n-\ell}(x)| \leq \frac{D\lambda}{m} + \frac{9}{7} P_1 + \frac{3}{2} Q_1, \quad m = 1, 2, \ldots.
\]
Now direct calculation shows that
\[
P_1 \leq \frac{(4H+1)\pi\lambda}{45H^{1/2}m}, \quad Q_1 \leq \frac{(2H+1)\pi\lambda}{5(2H)^{1/2}m}, \quad m = 2, 3, \ldots,
\]
the equalities holding for \( m = 2 \). Hence it appears that
\[
|f(x) - P_{\gamma_4m-\ell}(x)| \leq \frac{L\lambda}{m}, \quad m = 1, 2, 3, \ldots,
\]
where \( L \) depends on \( H \) only.

Now let \( n \) be an arbitrary integer, and let \( m = (n+4)/4, (n+3)/4, (n+2)/4, (n+1)/4 \), respectively, according as \( n \) has the form \( 4k \), \( 4k-1, 4k-2, 4k-3 \). In any event \( 4m - 4 \leq n < 4m, \gamma_n \leq nH \). Let the corresponding \( P_{\gamma_4m-\ell}(x) \) be denoted by \( P_{\gamma_n}(x) \), a polynomial of degree \( \gamma_n \). Since \( 1/m \leq 4/n \leq 4H/\gamma_n \) it follows that on \([0, 1]\)
\[
|f(x) - P_{\gamma_n}(x)| \leq \frac{4HL\lambda}{\gamma_n}.
\]
Consider the original sequence \( x^{\mu_0}, x^{\mu_1}, x^{\mu_2}, \ldots \). Let \( P_{\mu_n}(x) \) be a polynomial of degree \( \mu_n \) formed from this sequence, \( P_{\mu_n}(x) = P_{\gamma_k}(x) \) where \( \gamma_k \) is the largest of the \( \gamma \)'s in \( \mu_0, \mu_1, \ldots, \mu_n \). Now \( \mu_n \leq kH \), since otherwise \( \gamma_k \) would not be the largest \( \gamma \) in the given set of \( \mu \)'s. If \( k \geq 2 \), then \( 2\gamma_k > (2k-2)H \geq kH \geq \mu_n \), \( 1/\gamma_k < 2/\mu_n \), and \( |f(x) - P_{\mu_n}(x)| \leq 8HL\lambda/\mu_n \). If \( k = 0, 1 \) then \( P_{\mu_n}(x) = P_{\gamma_0}(x), \mu_n \leq H \), \( 1 \leq H/\mu_n \), and \( |f(x) - P_{\mu_n}(x)| \leq L\lambda/1 \leq H\lambda/\mu_n \). Hence the first of these two inequalities on \( f(x) \) serves for all \( n \), and the theorem is established.

This result may be extended to \([0, b]\), \( b > 0 \), by assuming that \( f(x) \) satisfies a Lipschitz condition \( H \), constant \( \lambda \), on \([0, b]\) and substituting \( x = by \). Then there exists \( P_{\mu_n}(y) \) of degree \( \mu_n \) such that on \([0, 1]\)
\[
|f(by) - P_{\mu_n}(y)| \leq \frac{KbH\lambda}{\mu_n},
\]
whence on \([0, b]\)
\[
|f(x) - P_{\mu_n}(x)| \leq \frac{KbH\lambda}{\mu_n},
\]
with \( P_{\mu_n}(x) = P_{\mu_n}(y) \).
The result may also be extended to $[a, b]$, where $a > 0$, by defining $f(x) = f(a)$ on $[0, a]$. In this case the condition on $f(x)$ may be replaced by one of the two simpler conditions.\(^6\)

**Theorem II.** Suppose $f(x)$ is continuous on $[0, 1]$ and let\(^8\) $\omega_H(\delta) = \max |f(x_2) - f(x_1)|$ for all $x_1, x_2$ on $[0, 1]$ satisfying the condition $|x_2 - x_1| \leq \delta$, and $\omega_H(0) = \omega_H(1)$ if $\delta > 1$. Then for each $\mu_n > 0$ there exists a polynomial $P_{\mu_n}(x)$ such that on $[0, 1]$

$$|f(x) - P_{\mu_n}(x)| \leq K\omega_H \left( \frac{1}{\mu_n} \right)$$

with $K = K + 2$.

Let

$$\phi(x) = a_ix^H + b_i, \quad \left( \frac{i}{\mu_n} \right)^{1/H} \leq x \leq \left( \frac{i + 1}{\mu_n} \right)^{1/H}, \quad i = 0, 1, \ldots,$$

$$a_i = \mu_n \left\{ f_1 \left[ \left( \frac{i + 1}{\mu_n} \right)^{1/H} \right] - f_1 \left[ \left( \frac{i}{\mu_n} \right)^{1/H} \right] \right\},$$

$$b_i = (i + 1)f_1 \left[ \left( \frac{i}{\mu_n} \right)^{1/H} \right] - if_1 \left[ \left( \frac{i + 1}{\mu_n} \right)^{1/H} \right],$$

where $f_1(x) = f(x)$, $[0, 1]$, $f_1(x) = f(1)$, $x \geq 1$. Then

$$|\phi(x_2) - \phi(x_1)| \leq \mu_n\omega_H \left( \frac{1}{\mu_n} \right) |x_2^H - x_1^H|, \quad [0, 1],$$

$$|f(x) - \phi(x)| \leq 2\omega_H \left( \frac{1}{\mu_n} \right), \quad [0, 1].$$

It follows from the first of these two inequalities that a $P_{\mu_n}(x)$ exists such that $|\phi(x) - P_{\mu_n}(x)| \leq K\omega_H(1/\mu_n)$ on $[0, 1]$, and the theorem follows. This theorem may also be extended to $[a, b]$, $0 \leq a \leq b$, in which case

$$|f(x) - P_{\mu_n}(x)| \leq K'\omega_H \left( \frac{b^H - a^H}{\mu_n} \right).$$

**Theorem III.** Let $\mu_n$ be given as before and let $\mu_n = H$ be the smallest of the $\mu$'s for which $\mu_{n+1} - \mu_n \leq \mu_a$ for all $n$. Suppose $f(x)$ has a derivative for every $x$ on $[a, b]$, and that $x^{1-H}f'(x)$ is continuous on $[a, b]$. Let $\eta_r = \mu_{r+a} - H$ for all $r \geq 0$. Suppose that there is given a sequence of positive numbers $\{\epsilon_n\}$, and suppose that for each $\epsilon_n$ there exists $P_n(x)$, a

---

\(^6\) Hereafter the function $\omega_H(\delta)$ will be called a “modulus of continuity $H$.”
polynomial in \( x^n \) of degree \( \eta_n - \alpha \), such that on \( [a, b] \) \( |x^{1-H}f'(x) - P_n(x)| \leq \epsilon_n \). Then, for every \( n > \alpha \) there exists \( P_{\mu_n}(x) \), a polynomial in \( x^n \) of degree \( \mu_n \), such that on \( [a, b] \)

\[
|f(x) - P_{\mu_n}(x)| \leq \frac{K'' \epsilon_n}{\mu_n}
\]

where \( K'' \) depends on \( H, \alpha, \) and \( b \).

Let

\[
R_n(x) = \int_a^x x^{H-1}P_n(x)\,dx,
\]

and let \( r_n(x) = f(x) - R_n(x) \). Then, with the possible exception of \( x = a \), \( r_n'(x) \) may be found; hence \( x^{1-H}r_n'(x) \), from which it follows that

\[
|r_n'(x)| \leq x^{H-1}\epsilon_n.
\]

Since

\[
r_n(x_2) - r_n(x_1) = \int_{x_1}^{x_2} r_n'(x)\,dx,
\]

it follows that on \( [a, b] \)

\[
|r_n(x_2) - r_n(x_1)| \leq \frac{\epsilon_n}{H} \left| x_2^H - x_1^H \right|.
\]

Therefore there exists \( \overline{R}_n(x) \) in \( x^n \) of degree \( \mu_n \) which approximates \( r_n(x) \) with error at most \( K\epsilon_n/H\mu_n \) and, with \( P_{\mu_n}(x) = R_n(x) + \overline{R}_n(x) \),

\[
|f(x) - P_{\mu_n}(x)| \leq \frac{K\epsilon_n}{H\mu_n}.
\]

From Theorem III it follows that if \( x^{1-H}f'(x) \) satisfies a Lipschitz condition \( H \), constant \( \lambda \), on \( [a, b] \), then for every \( n > \alpha \) there exists \( P_{\mu_n}(x) \) such that

\[
|f(x) - P_{\mu_n}(x)| \leq \frac{K\lambda}{\mu_n(\mu_n - H)}.
\]

If \( x^{1-H}f'(x) \) has a modulus of continuity \( H \) on \( [a, b] \), then for every \( n > \alpha \) there exists a polynomial \( P_{\mu_n}(x) \) of degree \( \mu_n \) for which

\[
|f(x) - P_{\mu_n}(x)| \leq \frac{K_2\omega_H}{\mu_n} \left( \frac{b^H - a^H}{\mu_n - H} \right).
\]

4. Uniform convergence. With the sequence \( \{x^{\mu_n}\} \) and a non-negative function \( \rho(x) \), integrable on \( [a, b] \) with positive integral, poly-
nomials \( p_{\mu_n}(x) \) in \( x^\mu \) of degree \( \mu_n \) can be constructed for every \( \mu_n \) by well known methods so that these polynomials will be orthonormal with respect to \( \rho(x) \) on \([a, b] \). If \( f(x) \) is an arbitrary continuous function, it follows as a formal result

\[
(1) \quad f(x) \sim \sum_{n=0}^{\infty} a_n p_{\mu_n}(x),
\]

where

\[
a_n = \int_a^b f(t) p_{\mu_n}(t) \rho(t) \, dt.
\]

In the following the results of the previous section are combined with certain results from the theory of orthogonal polynomials to give theorems on convergence for the above series expansion.\(^9\) It will be necessary at this point to suppose that the \( \mu \)'s are integers, since Bernstein's theorem is implicitly involved in the following. It is to be observed that this restriction admits sequences so irregular that the results obtained are by no means trivial, and further that the admission of any set of commensurable exponents would not be a material generalization.

In the following statement of those results from orthogonal polynomial theory\(^10\) which are applied to the present problem of convergence, all polynomials are polynomials in \( x^\mu \), of degree given by their subscript, and the use of \( \epsilon_n \) implies that there exists a polynomial \( P_{\mu_n}(x) \) such that on \([a, b] \), \( |f(x) - P_{\mu_n}(x)| \leq \epsilon_n \). If \( Q_{\mu_n}(x) \) is a polynomial minimizing\(^11\)

\[
\int_a^b \rho(x) \left[f(x) - Q_{\mu_n}(x)\right]^2 \, dx,
\]

if \( \rho(x) \) has a positive lower bound on \([a, b] \), then there are constants \( C_1, C_2 \), independent of \( x \) and \( n \), such that

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\(^9\) For the sequence \( 1, x, x^2, \ldots, x^{n-1}, x^n, x^{n+1}, \ldots, x^{\mu_n}, x^{\mu_n+1}, \ldots, x^{\mu_n+n-1}, \ldots \), where \( 0 < \mu_1 < \mu_2 < \cdots < \mu_{N-1} < \mu \), a recursion formula and hence a Christoffel-Darboux formula can be given for the orthonormal sequence. It is then possible to construct a theory for point-by-point convergence in the usual way. Since, on the one hand, there are no non-trivial results on boundedness of \( p_{\mu_n}(x) \); and, on the other hand, no very interesting extensions of Bernstein's and Markoff's theorems are available, the results which are obtainable are omitted.


\(^11\) It is a well known fact that \( Q_{\mu_n}(x) = \sum a_n p_{\mu_n}(x) \). Statements (A) and (B) therefore refer to the error of approximation by partial sums of the series for \( f(x) \).
(A) \[ | f(x) - Q_n(x) | \leq C_1 \mu_n \varepsilon_n, \quad a \leq x \leq b, \]
(B) \[ | f(x) - Q_n(x) | \leq C_2 \mu_n^{1/2} \varepsilon_n, \quad c \leq x \leq d, \]

where \( a < c < d < b \).

The problem of convergence then reduces to one of placing sufficient conditions on \( f(x) \) so that \( \mu_n \varepsilon_n \rightarrow 0 \) or \( \mu_n^{1/2} \varepsilon_n \rightarrow 0 \). It is assumed in the following theorems that \( p(x) \) is bounded from zero and integrable. From the results of the preceding sections values of \( \varepsilon_n \) are obtained which lead to the following theorems.

**Theorem IV.** If \( f(x) \) has a derivative for which \( x^{1-H}f'(x) \) satisfies a Lipschitz condition \( H \) on \( [a, b] \), then the series (1) converges uniformly to \( f(x) \) throughout \( [a, b] \), and the magnitude of the remainder after \( n \) terms does not exceed \( O[1/(\mu_n-H)] \).

**Theorem V.** If \( f(x) \) has a derivative for which \( x^{1-H}f'(x) \) is continuous on \( [a, b] \) and has a modulus of continuity \( H, \omega_H(\delta), (1) \) converges uniformly to \( f(x) \) throughout \( [a, b] \), and the magnitude of the remainder after \( n \) terms does not exceed \( O[\omega_H((b^H-a^H)/(\mu_n-H))] \).

**Theorem VI.** If \( f(x) \) satisfies a Lipschitz condition \( H \) on \( [a, b] \), (1) converges uniformly to \( f(x) \) throughout \( [c, d] \), where \( a < c < d < b \), and the magnitude of the remainder after \( n \) terms does not exceed \( O(1/\mu_n^{1/2}) \).

**Theorem VII.** If \( f(x) \) has a modulus of continuity \( H, \omega_H(\delta) \), for which \( \lim_{\delta \to 0} \omega_H(\delta)/\delta^{1/2} = 0 \), (1) converges uniformly to \( f(x) \) on \( [c, d] \), where \( c \) and \( d \) have the same meaning as before.

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