

position fields coincide and are cyclic. The field \bar{L} is then equivalent to a subfield of \bar{K}' ; without loss of generality we may suppose $\bar{K}' > \bar{L} \geq \bar{K}$. The degree $[\bar{L}:\bar{K}] = \bar{m}$ is a divisor of m . Consequently $[Z_n \bar{L}:\bar{K}] = [Z_n \bar{L}:\bar{L}][\bar{L}:\bar{K}] = n\bar{m}$. By the Galois theory there is then for every integer n an extension Z_n^* of degree n over \bar{K} . The defining equation $f^*(x) = 0$ of Z_n^*/\bar{K} now may be approximated by an irreducible equation $f(x) = 0$ of degree n with coefficients in K so that Z_n^* is generated by the roots of $f(x) = 0$. The root field of $f(x) = 0$ over K is the cyclic extension Z_n' of degree n over K . Hence $Z_n^* = Z_n' \bar{K}$ for all n , contrary to the assumption that K is not relatively complete with respect to any rank one valuation.

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A DIFFERENTIAL GEOMETRY PROBLEM USING TENSOR ANALYSIS

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1. **Introduction.** The problem at hand was worked out in attempting to apply tensors to a much more general problem in classical differential geometry. The results obtained in a general coordinate system reduce readily to classical results of Eisenhart. An interesting interpretation of Christoffel symbols appears.

2. **R net.** A rectilinear congruence in 3-space is called a W -congruence if the asymptotic lines on the two focal surfaces correspond. If the tangents to both families of curves of a conjugate net on a surface form W -congruences the net is called an R net.¹ We derive the analytic conditions that must obtain in order that a given conjugate net on a surface shall be an R net.

3. **Equations for an R net.** Let S_1 be one focal surface of a W -congruence, the vector equation of the surface being

$$(3.1) \quad z_1^\alpha = z_1^\alpha(x^i), \quad \alpha = 1, 2, 3; i = 1, 2.$$

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¹ Tzitzeica, Comptes Rendus de l'Académie des Sciences, Paris, vol. 152 (1911), p. 1077.

Let S_2 be the other focal surface of the congruence with vector z_2^α so that we have

$$(3.2) \quad z_2^\alpha = z_1^\alpha + \rho_1 \xi_1^\alpha$$

where ξ_1^α is a unit vector tangent to S_1 , and ρ_1 is an invariant.

Then if $\lambda_{1/i}^\alpha$ are the components of this vector in the x 's we have

$$(3.3) \quad \xi_1^\alpha = z_{1/i}^\alpha \lambda_{1/i}^\alpha$$

where in this case

$$z_{1/i}^\alpha = \frac{\partial z_1^\alpha}{\partial x^i}$$

since z_1^α being an invariant for a transformation of coordinates in the x 's, the ordinary derivative of z_1^α with respect to x^i is the same as the covariant derivative with respect to g_{ij} , the fundamental tensor of S_1 . Then substituting (3.3) in (3.2) we have

$$(3.4) \quad z_2^\alpha = z_1^\alpha + \rho_1 z_{1/i}^\alpha \lambda_{1/i}^\alpha$$

Similarly by the property of focal surfaces we have

$$(3.5) \quad z_1^\alpha = z_2^\alpha + \rho_2 z_{2/i}^\alpha \lambda_{2/i}^\alpha$$

where ρ_2 is an invariant, and $\lambda_{2/i}^\alpha$ is a unit vector tangent to S_2 .

Adding (3.4) and (3.5) we have

$$(3.6) \quad \rho_1 \lambda_{1/i}^\alpha z_{1/i}^\alpha + \rho_2 \lambda_{2/i}^\alpha z_{2/i}^\alpha = 0.$$

From (3.6) we have

$$(3.6') \quad \begin{aligned} \rho_1 &= \bar{e} \rho_2, \\ \lambda_{1/i}^\alpha z_{1/i}^\alpha &= \bar{e} \lambda_{2/i}^\alpha z_{2/i}^\alpha \end{aligned}$$

where $\bar{e} = 1$ if $e = -1$, and $\bar{e} = -1$ if $e = 1$, and conversely.

We differentiate (3.4) covariantly and have

$$(3.7) \quad z_{2/i,k}^\alpha = z_{1/i,k}^\alpha + \rho_{1,i} \lambda_{1/i}^\alpha z_{1/i}^\alpha + \rho_{1,i} \lambda_{1/i,k}^\alpha z_{1/i}^\alpha + \rho_{1,i} \lambda_{1/i}^\alpha z_{1/i,k}^\alpha.$$

Let η_1^α be the unit normal to S_1 .

We multiply (3.7) by $\lambda_{2/j}^k$, sum for k , multiply by η_1^α , sum for α and we have

$$(3.8) \quad 0 = \rho_1 b_{1/i;j} \lambda_{1/i}^\alpha \lambda_{2/j}^\alpha$$

the first three terms on the right vanishing because η_1^α is perpendicular to S_1 , the term on the left vanishing because of this fact and (3.6).

and the last term becoming $\rho_1 \eta_1^\alpha \cdot z_{i,j}^\alpha \lambda_{1,i}^i \lambda_{2,j}^j$ the second factor of which is denoted by $b_{1/i,j} \lambda_{1,i}^i \lambda_{2,j}^j$.²

Hence the directions $\lambda_{1,i}^i$ and $\lambda_{2,j}^j$ are conjugate on S_1 .³

Similarly they are conjugate on S_2 .

We next differentiate (3.7) covariantly with respect to g_{ij} , the fundamental tensor of S_1 ,

$$\begin{aligned}
 (3.9) \quad \frac{\partial^2 z_2^\alpha}{\partial x^k \partial x^j} - z_{2/m}^\alpha \left\{ \begin{matrix} m \\ kj \end{matrix} \right\} g_{ij} &= z_{1/k,j}^\alpha + \rho_{1/k,j} \lambda_{1,i}^i z_{1,i}^\alpha + \rho_{1/k} \lambda_{1,i}^i z_{1,i}^\alpha \\
 &+ \rho_{1/k} \lambda_{1,i}^i z_{1,i}^\alpha + \rho_{1,j} \lambda_{1/k}^i z_{1,i}^\alpha + \rho_{1,i} \lambda_{1/k}^i z_{1,i}^\alpha + \rho_{1,i} \lambda_{1,i}^i z_{1,i}^\alpha \\
 &+ \rho_{1/j} \lambda_{1,i}^i z_{1,i}^\alpha + \rho_{1,i} \lambda_{1/j}^i z_{1,i}^\alpha + \rho_{1,i} \lambda_{1,i}^i z_{1,i}^\alpha.
 \end{aligned}$$

Multiply by η_2^α (the unit normal to S_2), sum for α and we have

$$\begin{aligned}
 (3.10) \quad b_{2/kj} &= b_{1/k} \eta_1^\alpha \cdot \eta_2^\alpha + \rho_{1/k} \lambda_{1,i}^i \eta_1^\alpha \cdot \eta_2^\alpha + \rho_{1/k} \lambda_{1,i}^i \eta_1^\alpha \cdot \eta_2^\alpha \\
 &+ \rho_{1/k} \lambda_{1,i}^i b_{1/i} \eta_1^\alpha \cdot \eta_2^\alpha + \rho_{1,j} \lambda_{1/k}^i \eta_1^\alpha \cdot \eta_2^\alpha + \rho_{1,i} \lambda_{1/k}^i \eta_1^\alpha \cdot \eta_2^\alpha \\
 &+ \rho_{1,i} \lambda_{1/k}^i b_{1/i} \eta_1^\alpha \cdot \eta_2^\alpha + \rho_{1/j} \lambda_{1,i}^i b_{1/i} \eta_1^\alpha \cdot \eta_2^\alpha + \rho_{1,i} \lambda_{1,i}^i b_{1/i} \eta_1^\alpha \cdot \eta_2^\alpha \\
 &+ \rho_{1,i} \lambda_{1,i}^i b_{1/i} \eta_1^\alpha \cdot \eta_2^\alpha + b_{1/ik} \rho_{1,i} \eta_1^\alpha \cdot \eta_2^\alpha.
 \end{aligned}$$

In the future, since unless otherwise stated covariant differentiation is with respect to the fundamental tensor of S_1 , we shall note the covariant derivative of $z_{2/k}^\alpha$ by $z_{2/k,j}^\alpha$.

We evaluate $z_{1,i}^\alpha \cdot \eta_2^\alpha$ as follows.

We differentiate (3.5) covariantly giving

$$(3.11) \quad z_{1,i}^\alpha = z_{2/i}^\alpha + \rho_{2/i} \lambda_{2/s}^\alpha z_{2/s}^\alpha + \rho_{2/s} \lambda_{2/i}^\alpha z_{2/s}^\alpha + \rho_{2/s} \lambda_{2/s}^\alpha z_{2/s}^\alpha.$$

Multiply by η_2^α and sum for α , giving

$$(3.12) \quad z_{1,i}^\alpha \cdot \eta_2^\alpha = \rho_2 \lambda_{2/s}^\alpha b_{2/si}.$$

Substituting this value for $z_{1,i}^\alpha \cdot \eta_2^\alpha$ in (3.10) we have

$$\begin{aligned}
 (3.13) \quad b_{2/kj} &= (\eta_1^\alpha \cdot \eta_2^\alpha) (b_{1/kj} + b_{1/i} \rho_{1/k} \lambda_{1,i}^i + b_{1/i} \rho_{1,i} \lambda_{1/k}^i + b_{1/ik} \rho_{1,i} \lambda_{1,i}^i \\
 &+ b_{1/ik} \rho_{1,i} \lambda_{1,i}^i + b_{1/ik} \rho_{1,i} \lambda_{1,i}^i) + \rho_{1,i} \lambda_{1/i}^i b_{1/ik} \eta_1^\alpha \cdot \eta_2^\alpha \\
 &+ \rho_{1/k} \lambda_{1/i}^i \rho_{2/s} \lambda_{2/s}^\alpha b_{2/si} + \rho_{1/j} \lambda_{1/k}^i \rho_{2/s} \lambda_{2/s}^\alpha b_{2/si} \\
 &+ \rho_{1,i} \lambda_{1/k}^i \rho_{2/s} \lambda_{2/s}^\alpha b_{2/si}.
 \end{aligned}$$

Since the asymptotic lines on S_1 and S_2 are to correspond we have

² L. P. Eisenhart, *Riemannian Geometry*, 1926, Equation 56.2, p. 189.

³ *Ibid.*, Equation 56.3, p. 189.

$$(3.14) \quad b_{2/is} = \mu b_{1/is}.$$

We determine μ as follows. Differentiate the second of (3.6') co-variantly, giving

$$(3.15) \quad \lambda_{1/,j}^i \alpha_{z_{1/,i}} + \lambda_{1/z_{1/,ij}}^i \alpha + e(\lambda_{2/,j}^i \alpha_{z_{2/,i}} + \lambda_{2/z_{2/,ij}}^i \alpha) = 0.$$

Multiply by η_2^α , sum for α and use (3.12). This becomes

$$(3.16) \quad \lambda_{1/b_{1/,j}}^i (\eta_1^\alpha \cdot \eta_2^\alpha) + \rho_2 \lambda_{1/,j}^i \lambda_{2/b_{2/,ki}}^k + e \lambda_{2/b_{2/,ij}}^i = 0.$$

Now multiply by $\lambda_{1/}^j$ and sum for j giving

$$(3.17) \quad (\eta_1^\alpha \cdot \eta_2^\alpha) b_{1/,j} \lambda_{1/}^i \lambda_{1/}^j + b_{2/ik} \rho_2 \lambda_{1/,j}^i \lambda_{2/}^k \lambda_{1/}^j = 0.$$

But since $b_{2/ik} = \mu b_{1/ik}$ we have

$$(3.18) \quad \mu = - \frac{(\eta_1^\alpha \cdot \eta_2^\alpha) b_{1/,j} \lambda_{1/}^i \lambda_{1/}^j}{\rho_2 b_{1/ik} \lambda_{1/,j}^i \lambda_{2/}^k \lambda_{1/}^j}.$$

Substituting in (3.13) and using the fact that³

$$\eta_{1/,j}^\alpha = - b_{1/ej} g^{em} \alpha_{z_{1/,m}}$$

we have

$$(3.19) \quad \begin{aligned} & \rho_2 b_{is} \lambda_{2/}^s \lambda_{1/,h}^i \lambda_{1/}^h [b_{kj} + \rho_{1/,k} \lambda_{1/}^i b_{ij} + \rho_{1/} \lambda_{1/,k}^i b_{ij} + \rho_{1/,j} \lambda_{1/}^i b_{ik} \\ & + \rho_{1/} \lambda_{1/,j}^i b_{ik} + \rho_{1/} \lambda_{1/}^i b_{ik,j}] + b_{hr} \lambda_{1/}^h \lambda_{1/}^r [b_{kj} - \rho_{1/,k} \lambda_{1/,j}^i \rho_2 \lambda_{2/}^s b_{is} \\ & - \rho_{1/} \rho_2 \lambda_{1/,k}^i \lambda_{2/}^s b_{is} + b_{ik} \rho_1 \rho_2 \lambda_{1/}^i b_{e} g^{em} b_{ms} \lambda_{2/}^s \\ & - \rho_{1/,j} \lambda_{1/,k}^i \rho_2 \lambda_{2/}^s b_{is}] = 0 \end{aligned}$$

where the b 's are all those of S_1 .

To evaluate ρ_1 , multiply (3.16) by $\lambda_{2/}^j$ and sum for j , giving, by (3.14)

$$(3.20) \quad \rho_1 b_{1/ki} \lambda_{2/}^k \lambda_{1/,j}^i \lambda_{2/}^j + \lambda_{2/}^i \lambda_{2/}^j b_{1/ij} = 0.$$

Similarly, we have

$$(3.21) \quad \rho_2 b_{1/ki} \lambda_{1/}^k \lambda_{2/,j}^i \lambda_{1/}^j + \lambda_{1/}^i \lambda_{1/}^j b_{1/ij} = 0$$

where $\lambda_{2/,j}^i$ is with respect to \bar{g}_{ij} of S_2 . (It should be remarked that ρ_2 may be expressed entirely in terms of elements of S_1 by means by (3.6) and (3.7) and differentiation.)

Equations (3.19) with ρ_1 and ρ_2 determined by (3.20) and (3.21), respectively, constitute the condition that must obtain in order for

the tangents to the curves of direction $\lambda_{1/}^i$ on S_1 to form a W -congruence. An equation similar to (3.19) obtains for the direction $\lambda_{2/}^j$. These two equations must hold in order for the net with directions $\lambda_{1/}^i$ and $\lambda_{2/}^j$ to be an R net.

In particular we consider the case where $\lambda_{1/}^i$ and $\lambda_{2/}^j$ are tangent to the u and v parametric curves, respectively. Then⁴

$$(3.22) \quad \begin{aligned} \lambda_1^1 &= 1, & b_{11} &= D, \\ \lambda_1^2 &= 0, & b_{12} &= b_{21} = D' = 0, \\ \lambda_2^1 &= 0, & b_{22} &= D'', \\ \lambda_2^2 &= 1, \end{aligned}$$

and it is easily shown that

$$\lambda_{a/,i}^i = \begin{Bmatrix} i \\ aj \end{Bmatrix}$$

for any fixed i, a, j , which is an interesting interpretation of the Christoffel symbols in this case.

In this case (3.19) reduces to

$$(3.23) \quad 2 \frac{\partial}{\partial v} \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial u} \left(\begin{Bmatrix} 2 \\ 22 \end{Bmatrix} - \frac{D''}{D} \begin{Bmatrix} 2 \\ 11 \end{Bmatrix} \right),$$

the equation obtained by Eisenhart.⁵

The equation similar to (3.19) reduces to

$$(3.24) \quad 2 \frac{\partial}{\partial u} \begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial v} \left(\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} - \frac{D}{D''} \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} \right)$$

and these two equations constitute the condition that the parametric curves of a surface S form an R net.

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⁴ L. P. Eisenhart, *Differential Geometry*, 1909, p. 115.

⁵ L. P. Eisenhart, *Transformation of Surfaces*, 1923, p. 106.