ON THE NILPOTENCY OF THE RADICAL OF A RING

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1. **Introduction.** A few years ago, it was shown by C. Hopkins[2] that the structure theory of noncommutative rings[3] can be based on the assumption of only the minimum condition for left-ideals. Before Hopkins, a maximum condition for ideals had also been used in order to prove that the radical of the ring is nilpotent. Actually this last fact is a special case of the maximum condition, for example, the existence of a maximal nilpotent (two-sided) ideal, and this makes Hopkins' result appear rather surprising.

In this note, I give a short and simple proof for Hopkins' theorem. I also show that it is sufficient to assume only the minimum condition for sets of two-sided nil-ideals (that is, ideals consisting only of nilpotent elements) in order to prove the nilpotency of the radical. The later sections are concerned with the existence of idempotents and primitive left-ideals contained in a given regular left-ideal. Here the assumptions concerning the ring R are those on which Köthe[4] and Deuring[5] based their treatment of noncommutative rings. As was shown by Köthe, these assumptions are equivalent to the validity of the structure theory, so that it is natural to work with them. Once the results of the later sections have been established, there is no difficulty in developing the theory with the usual methods.[6]

2. **Preliminaries.** A ring R is a set of elements for which an addition and a multiplication are defined such that the elements form an abelian group under addition and that the associative law of multiplication and both distributive laws hold. We may also have a set K of operators. Then the product $t\alpha = \alpha t$ of any α in R with any t in K must be defined as an element of R, and the following rules are to hold (α, β in R, t in K)

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[1] Guggenheim Fellow.

[2] Charles Hopkins, Duke Mathematical Journal, vol. 4 (1938), p. 664; cf. also J. Levitzki, Compositio Mathematica, vol. 7 (1939), p. 214.

[3] E. Artin, Hamburger Abhandlungen, vol. 5 (1928), p. 251; B. L. van der Waerden, *Moderne Algebra*, vol. 2; M. Deuring, *Algebren*, Ergebnisse der Mathematik, vol. 4, 1935; A. A. Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939.

[4] G. Köthe, Mathematische Zeitschrift, vol. 32 (1930), p. 161.

[5] Loc. cit.

[6] The treatment thus obtained seems to me simpler than Deuring's treatment.

$$(1) \qquad (\alpha + \beta)t = \alpha t + \beta t, \qquad (\alpha\beta)t = \alpha(\beta t) = (\alpha t)\beta.$$

We then say that R is a K-ring. However, for some purposes, these postulates are not suitable, for example, it is easy to see that it is not always possible to imbed a K-ring R in a K-ring R^* which has a 1-element. We may modify the definition of a K-ring R in the following manner: If t lies in K and α lies in R, then αt and $t\alpha$ both are defined as elements of R. For α, β in R, and for t in K, we have

$$(2) \qquad \begin{gathered} (\alpha + \beta)t = \alpha t + \beta t, \qquad t(\alpha + \beta) = t\alpha + t\beta, \\ (\alpha\beta)t = \alpha(\beta t), \qquad (\alpha t)\beta = \alpha(t\beta), \qquad t(\alpha\beta) = t(\alpha\beta). \end{gathered}$$

We admit the possibility that $\alpha t \neq t\alpha$. A K-ring R in this sense can always be imbedded in a K-ring R^* which has a 1-element. It does not mean an essential restriction to assume that K itself is a ring which has a 1-element e such that: (a) $\alpha e = e\alpha = \alpha$ for all α in R. (b) If $\alpha t = 0$ for a fixed t in K and all α in R, then $t = 0$. The same holds, if all $t\alpha = 0$. (c) For the elements of R and for the elements of K, all possible associative and distributive laws hold. (This includes the equations (2).) A left-ideal (abbreviated l-ideal) of the K-ring R is a subset $\mathfrak{a}$ of R which satisfies the following conditions: (1) If α and β lie in $\mathfrak{a}$, then $\alpha \pm \beta$ lies in $\mathfrak{a}$. (2) If α lies in $\mathfrak{a}$, then $\rho\alpha$ and $t\alpha$ lie in $\mathfrak{a}$ for any ρ in R and any t in K. In the case of a right-ideal (r-ideal), (2) has to be replaced by: (2′) If α lies in $\mathfrak{a}$, then $\alpha\rho$ and αt lie in $\mathfrak{a}$ for any ρ in R and any t in K. A set $\mathfrak{a}$ is an ideal, if $\mathfrak{a}$ is both l-ideal and r-ideal.

For the following, it does not make any difference which definition of a K-ring is used.

3. **The radical.** An element ν of the ring R is a *radical-element*, if it belongs to at least one nilpotent ideal. Since every nilpotent l-ideal and every nilpotent r-ideal is contained in a nilpotent ideal,[7] the elements of nilpotent l-ideals and r-ideals are radical-elements. The sum of two nilpotent ideals is a nilpotent ideal;[8] the same holds for any finite number of nilpotent ideals. It follows readily that the set of all radical-elements forms an ideal N, the *radical* of R. It is easy to give examples of rings R whose radical N is not nilpotent. Hence we have to make a further assumption.

Assumption (A). *If Σ is a nonvacuous set of ideals $\mathfrak{a}$ which consist of nilpotent elements of R, then there exists at least one minimal ideal of Σ.*

[7] Cf. A. A. Albert, loc. cit., p. 22.
[8] Cf. A. A. Albert, loc. cit., p. 23.

We now prove this theorem.

THEOREM 1. *If the ring R satisfies this assumption* (A), *then its radical N is nilpotent.*

PROOF. (a) Let us first suppose that the ring R even satisfies the assumption (A) when the word "ideal" in it is replaced by the word "left-ideal."[9,10]

We have

$$N \supseteq N^2 \supseteq N^3 \supseteq \cdots . \tag{3}$$

Since all these ideals consist of nilpotent elements of R, there exists a minimal ideal $N^k = T$ of the set (3). If $T=0$, we are finished. Assume $T \neq 0$. Then

$$T^2 = T. \tag{4}$$

Consider the set Σ of all l-ideals $\mathfrak{a}$ contained in T for which $T\mathfrak{a} \neq 0$. This set is not empty, since it contains $\mathfrak{a} = T$. Let $\mathfrak{a}$ be a minimal l-ideal of Σ. Since $T\mathfrak{a} \neq 0$, there exists an element α in $\mathfrak{a}$ such that $T\alpha \neq 0$. Then $T\alpha \subseteq \mathfrak{a} \subseteq T$ and $T(T\alpha) = T^2\alpha = T\alpha \neq 0$. Hence $T\alpha$ itself belongs to Σ. Since $\mathfrak{a}$ was minimal, we have

$$\mathfrak{a} = T\alpha. \tag{5}$$

In particular, the element α of $\mathfrak{a}$ belongs to $T\alpha$. We can find an element τ of T such that $\alpha = \tau\alpha$. This implies $\alpha = \tau\alpha = \tau^2\alpha = \tau^3\alpha = \cdots$. However, τ as an element of $T = N^k$ is nilpotent. Hence $\tau^l\alpha = 0$ for a suitable l, and we obtain $\alpha = 0$ which contradicts $T\alpha \neq 0$. This proves Theorem 1 under our present assumption.

(b) If we assume that R satisfies the assumption (A) in its original form, we have to replace the set Σ by the set Σ' of all ideals $\mathfrak{a}$ contained in T for which $T\mathfrak{a}T \neq 0$. Again, the ideal T belongs to the set. If $\mathfrak{a}$ is a minimal ideal of Σ', we can find an element α of $\mathfrak{a}$ such that $T\alpha T \neq 0$. Then $T\alpha T$ belongs to Σ', and the minimal property of $\mathfrak{a}$ gives

$$\mathfrak{a} = T\alpha T. \tag{6}$$

Consequently, the element α of $\mathfrak{a}$ belongs to $T\alpha T$. This means that there exist elements $\tau_1, \tau_2, \cdots, \tau_n, \tau_1', \tau_2', \cdots, \tau_n'$ in T such that

[9] For the proof of the theorem, it is not necessary to deal with this case separately. However, the proof becomes somewhat simpler when we make the stronger assumption. The minimum condition for l-ideals of R, implies this stronger assumption.

[10] Added July 5, 1942: The proof in (a) was found independently by Reinhold Baer.

$$\alpha = \sum_{i=1}^{n} \tau_i \alpha \tau_i' .$$

On replacing α on the right side by $\sum \tau_j \alpha \tau_j'$ and continuing in this manner, we obtain

$$(7) \qquad \alpha = \sum_{i} \tau_i \alpha \tau_i' = \sum_{i,j} \tau_i \tau_j \alpha \tau_j' \tau_i' = \sum_{i,j,k} \tau_i \tau_j \tau_k \alpha \tau_k' \tau_j' \tau_i' = \cdots .$$

The radical element τ_i belongs to a nilpotent ideal $\mathfrak{n}_i$. Hence the sum $\mathfrak{q}$ of the n ideals $\mathfrak{n}_i$ is a nilpotent ideal containing all τ_i. If $\mathfrak{q}^r = 0$, then the rth of the sums in (7) vanishes since all products of r factors τ_i $(1 \leqq i \leqq n)$ will vanish. Hence $\alpha = 0$, which contradicts the condition $T\alpha T \neq 0$. This proves the theorem.

4. **Existence of idempotents.** For the last two sections, we make the following assumptions concerning the ring R:

(I) *The radical N of R is nilpotent.*

(II) *If Σ is a nonvacuous set of l-ideals $\mathfrak{a} \supseteq N$, there exists at least one minimal l-ideal of Σ.*

The condition (A), §3, implies the condition (I) as is shown by Theorem 1. If R satisfies the minimum condition for l-ideals, then certainly (A) and (I) hold, that is, (I) and (II) hold.

We say that an l-ideal is *regular*, if it is not nilpotent. An l-ideal $\mathfrak{a}$ is *primitive*, if $\mathfrak{a}$ is regular while every l-ideal $\mathfrak{b}$ with $\mathfrak{b} \subset \mathfrak{a}$ is nilpotent.

LEMMA 1. *Every regular l-ideal $\mathfrak{m}$ contains an element η with $\eta^2 \equiv \eta$, $\eta \not\equiv 0$* (mod N).

PROOF.[11] (a) Assume first that $\mathfrak{m} \supset N$. Using the assumption (II), we obtain an l-ideal $\mathfrak{a}$ with $\mathfrak{m} \supseteq \mathfrak{a} \supset N$ such that no l-ideal lies between $\mathfrak{a}$ and N. If $\mathfrak{a}\alpha \subseteq N$ for all α in $\mathfrak{a}$, we have $\mathfrak{a}^2 \subseteq N$ which would imply that $\mathfrak{a}^2$ is nilpotent. But then $\mathfrak{a}$ is nilpotent, that is, $\mathfrak{a} \subseteq N$. Hence for a suitable α in $\mathfrak{a}$, the l-ideal $\mathfrak{a}\alpha$ does not belong to N. Then $N \subset N + \mathfrak{a}\alpha \subseteq \mathfrak{a}$.[12] It follows that

$$\mathfrak{a} = N + \mathfrak{a}\alpha.$$

This implies that α can be written in the form $\alpha = \nu + \eta\alpha$ with ν in N and η in $\mathfrak{a}$. Then $\eta\alpha \equiv \alpha$ (mod N), and hence $\eta^2\alpha \equiv \eta\alpha$, $(\eta^2 - \eta)\alpha \equiv 0$

[11] If the minimum condition for l-ideals is satisfied in R, this proof can be simplified as follows: The l-ideal $\mathfrak{m}$ contains a primitive l-ideal $\mathfrak{a}$. As in the proof, we may choose an α in $\mathfrak{a}$ such that $\mathfrak{a}\alpha$ does not lie in N. Then $\mathfrak{a}\alpha = \mathfrak{a}$. This gives the existence of an η in $\mathfrak{a}$ for which $\eta\alpha = \eta$. As in the proof, we can conclude $\eta^2 \equiv \eta \not\equiv 0$ (mod N).

[12] We use the $+$ sign, even if the sum of the l-ideals is not direct.

(mod N). The elements x of $\mathfrak{a}$ for which $x\alpha\equiv 0$ (mod N) form an l-ideal $\mathfrak{b}$ with $N\subseteq\mathfrak{b}\subseteq\mathfrak{a}$. However, η does not lie in $\mathfrak{b}$, since $\eta\alpha\equiv 0$ (mod N) would imply $\alpha\equiv 0$ (mod N) and $\mathfrak{a}\alpha\subseteq N$. Hence $\mathfrak{b}\neq\mathfrak{a}$, that is, $\mathfrak{b}=N$. The element $\eta^2-\eta$ lies in $\mathfrak{b}$, which gives $\eta^2\equiv\eta$ (mod N). If we had $\eta\equiv 0$ (mod N), then again $\alpha\equiv\eta\alpha\equiv 0$ (mod N), which was impossible. Hence $\eta\not\equiv 0$, $\eta^2\equiv\eta$ (mod N) and η lies in $\mathfrak{a}$.

(b) If $\mathfrak{m}$ does not contain N, set $\mathfrak{m}^*=\mathfrak{m}+N$. Then, as shown in (a), the l-ideal $\mathfrak{m}^*$ contains an element η^* with $\eta^{*2}\equiv\eta^*\not\equiv 0$ (mod N). However, every η^* of $\mathfrak{m}+N$ is congruent to an element η of $\mathfrak{m}$, and this η will satisfy the conditions of Lemma 1.

LEMMA 2. *If r is a given positive integer, we may find a polynomial $f(x)$ with rational integral coefficients such that*

$$f(x) \equiv 0 \ (\text{mod } x^{r+1}), \qquad f(x) \equiv 1 \ (\text{mod } (1-x)^r). \tag{8}$$

PROOF. Expand the square bracket on the right side of $1=1^{2r}=[x+(1-x)]^{2r}$ according to the binomial theorem. If $f(x)$ is the sum of the terms containing x at least to the power x^{r+1}, then $f(x)$ satisfies the congruences (8).

THEOREM 2. *Every regular l-ideal $\mathfrak{m}$ contains an idempotent ϵ.*

PROOF. Construct η according to Lemma 1. Then $(\eta-\eta^2)^r=0$ for some r. The element $\epsilon=f(\eta)$ is well defined, as $f(x)$ has no constant term. It follows from (8) that we have an equation $f(x)^2-f(x)=(x-x^2)^r g(x)$ where $g(x)$ is a polynomial with rational integral coefficients such that $g(x)$ has no constant term. If we replace x by η, we obtain $\epsilon^2-\epsilon=0$. If we had $\epsilon=f(\eta)=0$, we could multiply the second congruence (8) by x^{r+1} and replace x by η. This would give $0=\eta^{r+1}$ which contradicts the congruences $\eta\equiv\eta^2\equiv\eta^3\equiv\cdots$, $\eta\not\equiv 0$ (mod N). Hence ϵ is an idempotent belonging to $\mathfrak{a}$.

COROLLARY. *An element ν of R is a radical element, if it is properly nilpotent, that is, if $\alpha\nu$ is nilpotent for every α in R.*

PROOF. If ν belongs to the nilpotent ideal $\mathfrak{n}$, then $R\nu\subseteq\mathfrak{n}\subseteq N$, and all $\alpha\nu$ are nilpotent. If ν is properly nilpotent, then $\mathfrak{a}=R\nu$ cannot contain an idempotent. Hence $R\nu\subseteq N$. The set of all ν for which $R\nu\subseteq N$, forms an ideal $\mathfrak{n}$ which again cannot contain an idempotent. Hence $\mathfrak{n}\subseteq N$; in particular, ν belongs to N.

5. **Primitive l-ideals contained in regular l-ideals.**[13] We prove the following theorems.

[13] If the minimum condition for l-ideals is assumed, Theorem 4 becomes trivial.

THEOREM 3. *Let* $\mathfrak{a}$ *be an l-ideal with* $\mathfrak{a}\supseteq N$, *such that no l-ideal lies between* $\mathfrak{a}$ *and* N. *If* ϵ *is an idempotent belonging to* $\mathfrak{a}$, *then* $R\epsilon$ *is a primitive l-ideal contained in* $\mathfrak{a}$.

PROOF. Suppose $\mathfrak{b}$ is a regular l-ideal with $\mathfrak{b}\subset R\epsilon$. Then $\mathfrak{b}$ contains an idempotent ϵ' and we have

$$R\epsilon' \subset R\epsilon. \tag{9}$$

Since ϵ' belongs to $R\epsilon$, we have $\epsilon'\epsilon=\epsilon'$. Set $\xi=\epsilon-\epsilon\epsilon'$. Then $\xi\epsilon=\epsilon^2-\epsilon\epsilon'\epsilon=\epsilon-\epsilon\epsilon'=\xi$, $\xi\epsilon'=0$. Hence $\xi^2=\xi\epsilon-\xi\epsilon\epsilon'=\xi$.

If $\xi\neq 0$, it is an idempotent contained in $R\epsilon$. Then $N\subseteq R\xi+N\subseteq\mathfrak{a}$. Since no l-ideal lies between $\mathfrak{a}$ and N and $R\xi$ contains $\xi^2=\xi\not\equiv 0$ (mod N), we have $R\xi+N=\mathfrak{a}$. This implies $\mathfrak{a}\epsilon'=R\xi\epsilon'+N\epsilon'=N\epsilon'\subseteq N$. However, $\mathfrak{a}\epsilon'$ contains $\epsilon'^2=\epsilon'$ which does not lie in N; we have a contradiction.

Hence $\xi=0$, that is, $\epsilon\epsilon'=\epsilon$. Then $R\epsilon'$ contains $\epsilon\epsilon'=\epsilon$, and $R\epsilon'\supseteq R\epsilon$. This contradicts (9), and the theorem is proved.

THEOREM 4. *Every regular l-ideal* $\mathfrak{m}$ *contains a primitive l-ideal.*

PROOF. Let $\mathfrak{a}$ be an l-ideal such that $N\subset\mathfrak{a}\subseteq\mathfrak{m}+N$ and that no l-ideal lies between N and $\mathfrak{a}$. Then $\mathfrak{a}$ contains an idempotent ϵ_0, and $\epsilon_0=\eta+\nu$ with η in $\mathfrak{m}$ and ν in N. Hence $\eta^2\equiv\epsilon_0^2=\epsilon_0\equiv\eta$ (mod N), $\eta\equiv\epsilon_0\not\equiv 0$ (mod N). Using Lemma 2 as in the proof of Theorem 2, we obtain an idempotent $\epsilon=f(\eta)$ which belongs to $\mathfrak{m}$. Then $R\epsilon\subseteq\mathfrak{m}$. Since $\eta=\epsilon_0-\nu$ lies in $\mathfrak{a}+N=\mathfrak{a}$, the element $\epsilon=f(\eta)$ lies in $\mathfrak{a}$. Theorem 3 shows that $R\epsilon$ is primitive.

We can now prove this theorem.

THEOREM 5. *Every l-ideal* $\mathfrak{m}$ *is a direct sum of primitive l-ideals* $R\epsilon_i$ *and a nilpotent l-ideal* $\mathfrak{n}$:

$$\mathfrak{m} = R\epsilon_1 + R\epsilon_2 + \cdots + R\epsilon_n + \mathfrak{n}.^{14} \tag{10}$$

Here the ϵ_i *can be taken as idempotents such that*

$$\epsilon_i\epsilon_j = 0 \quad for \quad i \neq j, \qquad \epsilon_i^2 = \epsilon_i, \qquad \mathfrak{n}\epsilon_i = 0. \tag{11}$$

PROOF. Because of the assumption (II), §4, we may assume that the theorem is correct for all regular r-ideals $\mathfrak{m}'$ with $\mathfrak{m}'+N\subset\mathfrak{m}+N$. Let $R\epsilon_n$ be a primitive l-ideal contained in $\mathfrak{m}$, ϵ_n an idempotent, and apply the Peirce decomposition. Then $\mathfrak{m}$ is a direct sum

$$\mathfrak{m} = \mathfrak{m}' + R\epsilon_n \tag{12}$$

[14] If $\mathfrak{m}$ is nilpotent, the terms $R\epsilon_i$ are missing.

where $\mathfrak{m}'$ consists of those elements μ of $\mathfrak{m}$ for which $\mu\epsilon_n=0$. This implies $\mathfrak{m}'+N\subset\mathfrak{m}+N$, as $(\mathfrak{m}'+N)\epsilon_n=N\epsilon_n\subseteq N$ while $(\mathfrak{m}+N)\epsilon_n$ contains ϵ_n. Then Theorem 5 holds for $\mathfrak{m}'$. If $\mathfrak{m}'=R\epsilon_1+\cdots+R\epsilon_{n-1}+\mathfrak{n}$ is the corresponding representation, then (12) gives the representation (10) of $\mathfrak{m}$. However, we obtain the formula (11), $\epsilon_i\epsilon_j=0$, only for $i\neq n$. We must replace ϵ_i $(i=1, 2, \cdots, n-1)$ by $\epsilon_i-\epsilon_n\epsilon_i$ in order to have $\epsilon_n\epsilon_i=0$. As is easily seen, these new elements satisfy all the conditions.

THEOREM 6. *If Theorem 5 is applied to* $R=\mathfrak{m}$, *then* $\zeta=\epsilon_1+\epsilon_2+\cdots+\epsilon_n$ *is a* 1-*element* (mod N), *that is*, $\alpha\zeta\equiv\zeta\alpha\equiv\alpha$ (mod N) *for all* α *in* R. *If* R *has a* 1-*element* 1, *then* $\zeta=1$, *and in the representation* (10) *of* $\mathfrak{m}=R$ *no term* $\mathfrak{n}$ *appears.*

PROOF. If we represent an element μ of R in accordance with (10) for $\mathfrak{m}=R$ we obtain easily from (11) that $\mu\zeta\equiv\mu$ (mod N). For any α in R, we then have $\mu(\zeta\alpha-\alpha)\equiv 0$ (mod N) for every μ in R. Consequently, $\zeta\alpha-\alpha$ is properly nilpotent, that is, $\zeta\alpha\equiv\alpha$ (mod N). If R contains a 1-element 1, then $\mu(1-\zeta)\equiv\mu-\mu\equiv 0$ (mod N) which proves that $1-\zeta$ is properly nilpotent. Since $(1-\zeta)^2=1-\zeta-\zeta+\zeta^2=1-\zeta$, the element $1-\zeta$ is either 0 or an idempotent. The latter case is excluded, hence $\zeta=1$. Finally, $\mu=\mu 1=\mu\zeta=\sum\mu\epsilon_i$ which shows that no term $\mathfrak{n}$ appears in this case.

Theorems 5 and 6 form the basis for the structure theory of rings, and for the theory of representations of rings.

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