METHODS OF SUMMATION

This seems to be the generalization of the classical result that a necessary and sufficient condition for the polar components of a matrix \( A \) to be commutative is that \( A \) be a normal matrix.

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REMARKS ON REGULARITY OF METHODS OF SUMMATION

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A doubly infinite matrix \( (a_{mn}) \) \((m, n = 1, 2, \cdots)\) is said to be regular, if for every sequence \( x = \{x_n\} \) with limit \( x' \) the corresponding sums \( y_m = \sum_{n=1}^{\infty} a_{mn} x_n \) exist for \( m = 1, 2, \cdots \), and if \( \lim_{m \to \infty} y_m = x' \). An apparently more inclusive definition of regularity is that for each sequence \( x \) with limit \( x' \) the sums defining \( y_m \) shall exist for all \( m \geq m_0(x) \) and \( \lim_{m \to \infty} y_m = x' \). Tamarkin\(^2\) has shown that \( (a_{mn}) \) is regular in the latter sense if and only if there exists an \( m_1 \) independent of \( x \) such that the matrix \( (a_{mn}) \) \((m \geq m_1, \ n \geq 1)\) is regular in the former sense. Using point set theory in the Banach space \( (c) \), he proves a theorem\(^3\) from which follows the result just mentioned. This note presents an elementary proof of that theorem and discusses some related topics.

**Theorem 1.** Suppose the doubly infinite matrix \( (a_{mn}) \) has the property that for each sequence \( x = \{x_n\} \) with limit 0 there exists an \( m_0 = m_0(x) \) such that for all \( m \geq m_0(x) \), \( u_m = \limsup_{k \to \infty} |\sum_{n=1}^{k} a_{mn} x_n| < \infty \). Then there exists an \( m_1 \) such that \( \sum_{n=1}^{\infty} |a_{mn}| < \infty \) for all \( m \geq m_1 \).

If in addition \( \lim_{m \to \infty} u_m = 0 \) for each sequence \( x \) with limit 0, it will follow\(^4\) that there exists an \( N \) such that \( \sum_{n=1}^{\infty} |a_{mn}| \leq N < \infty \), for all \( m \geq m_1 \).

To prove Theorem 1, suppose there were an infinite sequence \( m_1 < m_2 < \cdots \) such that \( \sum_{n=1}^{\infty} |a_{mn}| = \infty \) for \( m \in \{ m_r \} \). Let \( x_1, \cdots, x_{k_1} \) be chosen with unit moduli and with amplitudes such that

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1. Presented to the Society, April 11, 1942 under the title A remark on Toeplitz matrices; received by the editors January 22, 1942.
\[
\sum_{n=1}^{k_1} a_{mn}x_n = \sum_{n=1}^{k_1} a_{mn}x_n > 1.
\]

Let \( x_{k_1+1}, \ldots, x_{k_2} \) be chosen with moduli \( 1/2 \) and with amplitudes such that
\[
\left| \sum_{n=k_1+1}^{k_2} a_{mn}x_n \right| > 2 + \sum_{n=1}^{k_1} \left| a_{mn}x_n \right|.
\]

Let \( x_{k_2+1}, \ldots, x_{k_3} \) be chosen with moduli \( 1/3 \) and with amplitudes such that
\[
\left| \sum_{n=k_2+1}^{k_3} a_{mn}x_n \right| > 3 + \sum_{n=1}^{k_2} \left| a_{mn}x_n \right|.
\]

Writing \( y_m(k) = \sum_{n=1}^{k} a_{mn}x_n \), the sequence \( \{x_n\} \) and integers \( k_1 < k_2 < \cdots \) are thus chosen successively so that \( |y_{m_1}(k_1)| > 1 \), \( |y_{m_2}(k_2)| > 2 \), \( |y_{m_3}(k_3)| > 3 \), \( |y_{m_4}(k_4)| > 4 \), \( |y_{m_5}(k_5)| > 5 \), \( |y_{m_6}(k_6)| > 6 \), \( \cdots \); while \( |x_n| = 1/r \), for \( k_r - 1 < n \leq k_r \). This is a sort of alternating or "sweeping-out" process. So defined, \( \{x_n\} \) is a sequence with limit 0, but \( \limsup_{k \to \infty} \left| \sum_{n=1}^{k} a_{mn}x_n \right| = \infty \), for \( m \in \{m_r\} \). This contradiction completes the proof of Theorem 1.

The matrix \((a_{mn})\) is said to be null-preserving, if for every sequence \( x = \{x_n\} \) with limit 0 the corresponding sums defining \( y_m \) exist for \( m = 1, 2, \cdots \) and if \( \lim_{m \to \infty} y_m = 0 \). An apparently more inclusive definition of null-preserving is that for each sequence \( x \) with limit 0 we have \( u_m = \limsup_{k \to \infty} \left| \sum_{n=1}^{k} a_{mn}x_n \right| < \infty \) for all \( m \geq m_0(x) \) and \( \lim_{m \to \infty} u_m = 0 \). We remark that it is a consequence of Theorem 1 that \((a_{mn})\) is null-preserving in the latter sense if and only if there exists an \( m_1 \) such that the matrix \((a_{mn})\) \((m \geq m_1, n \geq 1)\) is null-preserving in the former sense.\(^6\)

To consider a problem which is related to the above in the method of proof, let each element of a matrix \((a_{mn})\) be either +1 or −1. For \( 0 \leq t \leq 1 \) and \( n = 1, 2, \cdots \) let \( \{\phi_n(t)\} \) be the Rademacher orthogonal functions,\(^6\) and let \( y_{mk}(t) = \sum_{n=1}^{k} a_{mn} \phi_n(t) \). Then it is well known\(^7\) that for almost all \( t \), for all \( m = 1, 2, \cdots \) and for all \( \epsilon > 0 \), \( \lim_{k \to \infty} k^{-1/2−\epsilon}y_{mk}(t) = 0 \). It is clear that for a particular fixed \( m \) there is a \( t \) such that \( \lim_{k \to \infty} k^{-1}y_{mk}(t) = 1 \). The problem is to show that there is a \( t \) such that


\(^7\) For references to this and more precise results, see A. Khintchine, *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik, Berlin, 1933, pp. 60–61.
simultaneously for all \( m = 1, 2, \ldots \), \( \lim \sup_{k \to \infty} k^{-1} y_{mk}(t) = 1 \). That there exists such a \( t \) can be shown by using the alternating process of Theorem 1.

Theorem 2 follows immediately from a theorem of Banach.⁸

**Theorem 2.** If \( E_m \) is a linear manifold satisfying Baire's condition⁹ in a Banach space \( E \) \((m = 1, 2, \ldots)\) and if \( \lim_{m \to \infty} E_m = E \), then there exists an \( m_1 \) such that \( E_{m_1} = E \).

Theorem 2 furnishes a Banach space analogue and a proof of Theorem 1 which is related to Tamarkin's proof. To see this, let \( E \) be the Banach space \((c_0)\) of sequences \( x = \{x_n\} \) convergent to 0, with \( \|x\| = \max_n |x_n| \), and with addition and multiplication by a (complex) scalar defined as usual. Let \((a_{mn})\) be as in Theorem 1. Let \( E_m \) be the subset of \( E \) for which \( \lim \sup_{k \to \infty} |\sum_{n=1}^{k} a_{rn} x_n| < \infty \) for all \( r \geq m \).

The hypotheses of Theorem 2 are satisfied, and from its conclusion it may be proved directly for an arbitrary \( m \geq m_1 \) that \( \sum_{n=1}^{\infty} |a_{mn}| < \infty \).


⁹ See S. Banach, loc. cit., p. 17. By considering a Hamel base for \( E \), G. W. Mackey has remarked to the authors that Theorem 2 is false if the words “satisfying Baire’s condition” are omitted.