

PROJECTIONS OF THE PRIME-POWER ABELIAN GROUP OF ORDER p^m AND TYPE $(m-1, 1)$

ROSS A. BEAUMONT

1. **Introduction.** A function f of the subgroups of the group G upon the subgroups of the group H is called a projectivity of G upon H ($f(G) = H$) if the following hold.

(1) For every subgroup S of G , $f(S)$ is a subgroup of H .

(2) If S' is a subgroup of H , then there exists a subgroup S of G such that $f(S) = S'$.

(3) If S and T are subgroups of G , $S \leq T$ is a necessary and sufficient condition that $f(S) \leq f(T)$.

The correspondence f is a (1-1) correspondence which preserves the partial ordering of the set of subgroups of the group G .

Further, a projectivity f is called index-preserving if $[T:S] = [f(T):f(S)]$ for subgroups S of cyclic subgroups T of G ; and f is called strictly index-preserving if $[T:S] = [f(T):f(S)]$ for subgroups S of subgroups T of G .

If G is the direct product of cyclic groups of order p , p a prime number, R. Baer¹ has given necessary and sufficient conditions that a group H be a projection of G . In particular he has shown that if the projectivity of G upon H is index-preserving, then G and H are isomorphic. Thus in a study of the projections of the prime-power abelian group of order p^m and type $(m-1, 1)$, we need consider only the case $m > 2$.

Rottlaender² investigated the case $m = 3$ and found necessary and sufficient conditions for the existence of a strictly index-preserving projectivity of the prime-power abelian group G of order p^3 and type $(2, 1)$ upon a group H .

In this note, Baer's general results are used to find the necessary and sufficient conditions for the existence of a projectivity of the prime-power abelian group G of order p^m and type $(m-1, 1)$ upon a group H .

2. **The necessary conditions.** If G is an abelian group of the type under consideration, $G = \{u_1\} \times \{u_2\}$ where u_1 is of order p^{m-1} , $m > 2$,

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¹ R. Baer, *The significance of the system of subgroups for the structure of the group*, American Journal of Mathematics, vol. 61 (1939), pp. 1-44. Hereafter this paper will be referred to as B.

² Ada Rottlaender, *Nachweis der Existenz nicht-isomorpher Gruppen von gleicher Situation der Untergruppen*, Mathematische Zeitschrift, vol. 28 (1928), pp. 641-653.

and u_2 is of order p . If $f(G) = H$, it follows from known results³ that f is index-preserving, and hence $H = \{f(\{u_1\}), f(\{u_2\})\}$ where $f(\{u_1\}) = \{u'_1\}$ is a cyclic group of order p^{m-1} , $f(\{u_2\}) = \{u'_2\}$ is a cyclic group of order p . Thus $H = \{u'_1, u'_2\}$ where u'_1 and u'_2 are independent generators of order p^{m-1} and p , respectively.

If K is any group, then K^p is the set of all p th powers of the elements of K , and K^p is a characteristic subset of K .

LEMMA 1. $f(G^p) = H^p$ is a characteristic subgroup of H .

PROOF. $G^p = \{u_1^p\}$ and $f(G^p) = f(\{u_1^p\}) = f(\{u_1\})^p = \{u_1'^p\}$ since f is index-preserving. $\{u_1'^p\} \leq H^p$. If x is in H^p , $x = y^p$ where y is in H . $f^{-1}(\{x\}) = f^{-1}(\{y^p\}) = f^{-1}(\{y\})^p = \{u_1^k u_2^m\}^p = \{u_1^{kp}\}$. $f(\{u_1^{kp}\}) = f(\{u_1\})^{kp} = \{u_1'^{kp}\} = \{x\}$. Hence $x = u_1'^{kp} = (u_1'^p)^{ik}$ and x is in $\{u_1'^p\}$. Thus $H^p \leq \{u_1'^p\}$ so that we have shown that $f(G^p) = \{u_1'^p\} = H^p$. Since H^p is a subgroup of H , it is a characteristic subgroup, and in particular normal.

Since $f(G) = H$ and $f(G^p) = H^p$, the index-preserving projectivity f of G upon H induces an index-preserving projectivity of G/G^p upon H/H^p . G/G^p is the direct product of two cyclic groups of order p and hence by the result⁴ mentioned above, G/G^p and H/H^p are isomorphic. Thus H/H^p is abelian and H^p contains the commutator subgroup of H . This implies

$$(1) \quad u_2'^{-1} u_1' u_2' = u_1'^{1+ip}$$

The following multiplication rules for elements in H follow from (1)

$$(2) \quad u_2'^{-1} u_1'^h u_2' = u_1'^{h(1+ip)}$$

$$(3) \quad u_1'^h u_2'^i = u_2'^i u_1'^{h(1+ip)^i}$$

$$(4) \quad (u_2' u_1')^k = u_2'^{ki} u_1'^{h[1+(1+ip)^i+\dots+(1+ip)^{(k-1)i}]}$$

$$= u_2'^{ki} u_1'^{h[(1+ip)^{ki}-1]/[(1+ip)^i-1]}$$

It follows from (3) that the order of H is p^m so that we have proved the following theorem.

THEOREM 1. If G is the prime-power abelian group of order p^m , $m > 2$, and type $(m - 1, 1)$ and if f is a projectivity of G upon a group H , then H is a prime-power group of order p^m generated by independent generators u'_1 of order p^{m-1} and u'_2 of order p such that $u_2'^{-1} u_1' u_2' = u_1'^{1+ip}$.

If $p = 2$, we derive the additional necessary condition.

³ [B, Corollary 11.3].

⁴ [B, Corollary 8.2].

LEMMA 2. $j \equiv 0 \pmod 2$ so that $u_2'^{-1}u_1' u_2' = u_1'^{1+4w}$.

PROOF. Since $f(\{u_1\}) = \{u_1'\}$, it follows from Baer's results⁵ that there exists one and only one element u' in H such that $f(\{u_2\}) = \{u'\}$ and $f(\{u_2u_1\}) = \{u'u_1'\}$. Since $\{u'\} = \{u_2'\}$, $u' = u_2'^\rho$ where ρ is odd. $f(\{u_2u_1\}^2) = f(\{u_2u_1\})^2$ since f is index-preserving and we have

$$\begin{aligned} f(\{u_2u_1\}^2) &= f(\{u_1^2\}) = f(\{u_1\})^2 = \{u_1'^2\}, \\ f(\{u_2u_1\})^2 &= \{(u'u_1')^2\} = \{(u_2'^\rho u_1')^2\} \\ &= \{u_2'^{2\rho} u_1'^{(1+2j)^{2\rho}-1} / [(1+2j)^{\rho}-1]\} = \{u_1'^{(1+2)^{\rho+1}}\}. \end{aligned}$$

Thus $u_1'^{(1+2j)^{\rho+1}} = u_1'^{2\gamma}$ where γ is odd and we have

$$\begin{aligned} (1 + 2j)^\rho + 1 &\equiv 2\gamma \pmod{2^{m-1}}, \\ 2 + 2\rho j + [\rho(\rho - 1)/2](2j)^2 + \dots &\equiv 2\gamma \pmod{2^{m-1}}, \\ 1 + \rho j + [\rho(\rho - 1)/2]2j^2 + \dots &\equiv \gamma \pmod{2^{m-2}} \end{aligned}$$

and recall that $m > 2$. Since γ is odd, the left member of the congruence is odd which implies ρj is even. Since ρ is odd, j is even, which completes the proof of the lemma.

3. Construction of a projectivity for groups satisfying the necessary conditions. If p is an odd prime and H is a group satisfying the necessary conditions, then H is either abelian or H is the unique non-abelian group $\{U_1, U_2\}$ where U_1 and U_2 are subject to the sole defining relations⁶

$$(5) \quad U_1^{p^{m-1}} = U_2^p = 1, \quad U_2^{-1}U_1U_2 = U_1^{1+p^{m-2}}.$$

If H is abelian, then G and H are isomorphic and this isomorphism induces a projectivity of G upon H .

If H is non-abelian we will construct a projectivity of G upon H by establishing a correspondence between the cyclic subgroups and then extending this correspondence to a projectivity.

From (5) we find the following multiplication rule for elements of H

$$(6) \quad (U_2^i U_1^h)^k = U_2^{ik} U_1^{h[k+ik(k-1)p^{m-2}/2]}.$$

Every element of G has the unique form $u_2^s u_1^t$ where $0 \leq s < p$, $0 \leq t < p^{m-1}$. If $s > 0$, there exists an integer x , uniquely determined

⁵ [B, (9.2), (d)].

⁶ Carmichael, *Introduction to the Theory of Groups of Finite Order*, p. 132.

mod p^{m-1} , such that $sx \equiv 1 \pmod{p^{m-1}}$, $(x, p) = 1$. Then $u_1^r = (u_1^t)^{sx} = u_1^{rx} = u_1^{ts}$ where $rx = t$. $u_2^s u_1^r = u_2^s u_1^{ts}$ and $\{u_2^s u_1^r\} = \{(u_2 u_1^t)^s\} = \{u_2 u_1^t\}$ where t is uniquely determined mod p^{m-1} . We define the following correspondence of the cyclic subgroups of G upon those of H

$$(7) \quad f(\{u_2 u_1^t\}) = \{U_2 U_1^t\},$$

$$(8) \quad f(\{u_1^r\}) = \{U_1^r\}.$$

The correspondence f is a (1-1) correspondence of the set of cyclic subgroups of G upon a subset of the set of cyclic subgroups of H . To show that f is a (1-1) correspondence on the whole set of cyclic subgroups of H , it is only necessary to show that every cyclic subgroup of H has the form $\{U_2 U_1^t\}$ or $\{U_1^r\}$ where t or r is uniquely determined mod p^{m-1} , respectively.

Every element of H has the unique form $U_2^s U_1^r$ where $0 \leq s < p$, $0 \leq r < p^{m-1}$. If $s > 0$, there exists an integer x , uniquely determined mod p^{m-1} , such that $sx \equiv 1 \pmod{p^{m-1}}$, $(x, p) = 1$. Then

$$\begin{aligned} \{U_2^s U_1^r\} &= \{(U_2^s U_1^r)^x\} \\ &= \{U_2^{sx} U_1^{r[x+(sx(x-1)/2)p^{m-2}]}\} \\ &= \{U_2 U_1^{r[x+(sx(x-1)/2)p^{m-2}]}\} \end{aligned}$$

where $r[x+(sx(x-1)/2)p^{m-2}]$ is uniquely determined mod p^{m-1} .

The correspondence f preserves the indices of the cyclic subgroups since

$$f(\{u_2 u_1^t\}^p) = f(\{u_1^{tp}\}) = \{U_1^{tp}\}$$

by (8),

$$f(\{u_2 u_1^t\})^p = \{U_2 U_1^t\}^p = \{U_1^{t[p+(p-1)/2]p^{m-2}}\} = \{U_1^{tp}\},$$

by (7) and (6).

Since the only non-cyclic subgroups of H are those of the form $\{U_1^{p^\lambda}, U_2\}$, $0 \leq \lambda < m-1$, by extending f so that

$$(9) \quad f(\{u_1^{p^\lambda}, u_2\}) = \{U_1^{p^\lambda}, U_2\}$$

f becomes an index-preserving projectivity of G upon H .

Combining the above results with Theorem 1 we have this theorem.

THEOREM 2. *If G is the prime-power abelian group of odd order p^m , $m > 2$, and type $(m-1, 1)$ then there exists a projectivity f of G upon a group H if, and only if, either H is isomorphic to G or H is the non-*

abelian group $\{U_1, U_2\}$ where U_1 and U_2 are subject to the sole defining relations (5).

If $p=2$ it follows from Lemma 2 that if $m=3$, then H is abelian and hence G and H are isomorphic groups. If $m>3$, it follows from Lemma 2 and from known results⁷ that H is either abelian or the non-abelian group $\{U_1, U_2\}$ where U_1 and U_2 are subject to the sole defining relations

$$(10) \quad U_1^{2^{m-1}} = U_2^2 = 1, \quad U_2 U_1 U_2 = U_1^{1+2^{m-2}}.$$

Since Baer has shown⁸ that there exists an index-preserving projectivity of G upon this non-abelian group we have the following theorem.

THEOREM 3. *If G is the prime-power abelian group of order 2^m , $m > 2$, and type $(m-1, 1)$ then*

(a) *if $m=3$, there exists a projectivity f of G upon a group H if and only if G and H are isomorphic groups;*

(b) *if $m > 3$, there exists a projectivity f of G upon a group H if, and only if, either G and H are isomorphic groups or H is the non-abelian group $\{U_1, U_2\}$ where U_1 and U_2 are subject to the sole defining relations (10).*

UNIVERSITY OF WASHINGTON

⁷ Carmichael, loc. cit., p. 133.

⁸ [B, p. 11].