THE MEASURE OF THE CRITICAL VALUES OF DIFFERENTIABLE MAPS

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1. Introduction. Consider the *map*

$$y^j = f^j(x^1, x^2, \cdots, x^m), \qquad j = 1, 2, \cdots, n, \tag{1.1}$$

of a region R of euclidean m-space into part of euclidean n-space. Suppose that each function f^j $(j=1, \cdots, n)$ is of class C^q in R $(q \geqq 1)$.[1] A *critical point* of the map (1.1) is a point in R at which the matrix of first derivatives $\mathfrak{M} = \|f_i^j\|$ $(i=1, \cdots, m; j=1, \cdots, n)$ is of less than maximum rank. The *rank* of a critical point x is the rank of $\mathfrak{M}$ at x. A *critical value* is the image under (1.1) of a critical point. If $n=1$, these definitions are the usual definitions of critical point and value of a continuously differentiable function.

We prove the following result: *If* $m \leqq n$, *the set of critical values of the map* (1.1) *is of* m*-dimensional measure*[2] *zero* without further hypothesis on q; *if* $m > n$, *the set of critical values of the map* (1.1) *is of* n*-dimensional measure zero providing that* $q \geqq m-n+1$. Using an example due to Hassler Whitney[3] we show that the hypothesis on q cannot be weakened. We prove also that the critical values of (1.1) corresponding to critical points of rank zero constitute a set of (m/q)-dimensional measure zero.

The idea of considering the measure of the set of critical values of one function or of several functions is due to Marston Morse.

The first result stated above reduces, if $n=1$, to the known theorem: The critical values of a function of m variables of class C^m constitute a set of linear measure zero. A. P. Morse[4] has given a proof of this theorem for all m. In the present paper we make use of one of A. P. Morse's results.

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[1] A function is of class C^q if all its partial derivatives of order q exist and are continuous.

[2] In the sense of Hausdorff-Saks. The definition is given in §2.

[3] H. Whitney, *A function not constant on a connected set of critical points*, Duke Mathematical Journal, vol. 1 (1935), pp. 514–517.

[4] A. P. Morse, *The behaviour of a function on its critical set*, Annals of Mathematics, (2), vol. 40 (1939), pp. 62–70. Proofs for the cases $m=1, 2, 3$ had previously been given by M. Morse and for the cases $m=4, 5, 6$ by M. Morse and the author in unpublished papers.

In a study of functional dependence,[5] A. B. Brown has shown that the set of critical values of (1.1) corresponding to a closed bounded set of critical points is nowhere dense providing that q satisfies certain conditions. The theorems of the present paper imply this particular result.

2. **s-dimensional measure.** Given a set A and positive quantities s and α. Denote the diameter of A by $\delta(A)$. Let $\{A_1, A_2, \cdots\}$ be a covering of A by sets of diameter less than α; and let $L_s(A;\alpha)$ be the greatest lower bound of the sums $\sum_k [\delta(A_k)]^s$ for all such coverings. The *s-dimensional outer measure* of A is

$$(2.1)\qquad L_s(A) = c_s \lim_{\alpha\to 0} L_s(A;\alpha), \qquad c_s = \pi^{s/2}/2^s\Gamma[(s+2)/2],$$

where c_s is, for integral s, the s-dimensional volume of a sphere of unit diameter in s-space.[6]

We shall say that A is an *s-null* set if $L_s(A)=0$. The value of c_s in (2.1) is immaterial to this paper as we are concerned with the nullity of sets. An s-null set is a fortiori an $(s+p)$-null set (p positive). $L_s(A)$ is a regular Carathéodory measure. In n-space $L_n(A)=|A|$, where $|A|$ is the outer Lebesgue measure of A.[7]

3. **Critical points of positive rank: change of variables.** Consider a critical point x_0 of rank $r>0$. Suppose (without real loss of generality) that the determinant $\Delta = |f_c^d|$ $(c, d=1,\cdots,r)$ is not zero at x_0. Consider the change of variables from x to u defined by the equations

$$(3.1)\qquad u^d = f^d(x), \quad u^{r+g} = x^{r+g}, \quad d = 1,\cdots,r;\ g = 1,\cdots,m-r.$$

Let u_0 be the image of x_0 under (3.1); and let J be the Jacobian matrix of (3.1). Since $|J| = \Delta \neq 0$ at x_0, the inverse of (3.1)

$$x^i = \phi^i(u), \qquad i = 1,\cdots,m,$$

exists and is of class C^q near u_0. In terms of the new variables u the map (1.1) is

$$\begin{aligned}&(3.2)\qquad &\begin{cases} y^d = u^d = F^d(u), & d = 1,\cdots,r,\\ y^{r+h} = f^{r+h}[\phi(u)] = F^{r+h}(u), & h = 1,\cdots,n-r.\end{cases}\\ &(3.3)\end{aligned}$$

Let $\mathfrak{M}'$ be the functional matrix $\|F_i^j\|$ $(i=1,\cdots,m;\ j=1,\cdots,n)$.

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[5] A. B. Brown, *Functional dependence*, Transactions of this Society, vol. 38 (1935), pp. 379–394.

[6] F. Hausdorff, *Dimension und äusseres Mass*, Mathematische Annalen, vol. 79 (1919), pp. 157–179. S. Saks, *Theory of the Integral*, Warsaw, 1937, pp. 53–54.

[7] We make use of this fact for the case $|A|=0$ only.

Then $\mathfrak{M}=\mathfrak{M}'J$ and therefore $\mathfrak{M}$ and $\mathfrak{M}'$ have the same rank since $|J|\neq 0$. Hence *we may consider* (3.2, 3.3) *near* u_0 *instead of* (1.1) *near* x_0 *without changing either critical values or ranks of critical points.*

4. **Case I:** $m\leqq n$. We prove Theorem 4.1.

THEOREM 4.1. *The critical values of the map* (1.1) *constitute an m-null set if* $m\leqq n$.

PROOF. We consider first the set A of critical points of rank zero. We shall show that there is a neighborhood N of each point of A such that $f(NA)$, the image of NA under (1.1), is an m-null set. As A can be covered by denumerably many such neighborhoods N it will follow that $f(A)$ is an m-null set.

Let N be an open m-cube[8] whose closure is in R. Suppose that x_1 is a point of NA and x_2 is a point of N. Let $y_e=f(x_e)$ $(e=1, 2)$. Then by the mean value theorem,

$$y_2^j - y_1^j = \sum_{i=1}^{m} \zeta_i^j(x_2^i - x_1^i), \qquad j = 1, \cdots, n, \tag{4.2}$$

where $\zeta_i^j\to 0$ as $x_2\to x_1$.

Let $C(\gamma)$ be a closed m-cube of side γ. It follows from (4.2) applied twice and the triangle inequality that if $C(\gamma)$ contains a point of A, then $\delta\{f[NC(\gamma)]\}\leqq 2m^2\zeta\gamma$ and $\delta\{f[NC(\gamma)]\}^m\leqq(2m^2\zeta)^m|C(\gamma)|$, where ζ is the least upper bound of the functions $|\zeta_i^j|$ for all x_1 in NA, x_2 in N such that $|x_2^i-x_1^i|\leqq\gamma$ $(i=1, \cdots, m; j=1, \cdots, n)$. Then $\zeta\to 0$ with γ.

Therefore given $\alpha>0$ and $\epsilon>0$, there exists a $G>0$ such that if $C(\gamma)$ contains a point of A, $\gamma<G$ implies that

$$\delta\{f[NC(\gamma)]\} < \alpha, \qquad \delta\{f[NC(\gamma)]\}^m < \frac{\epsilon}{|N|+1}|C(\gamma)|. \tag{4.3}$$

Consider the set of all cubes $C(\gamma)$ centered at points of NA and such that $\gamma<G$. By a theorem related to Vitali's covering theorem,[9] there exists a sequence $\{C_1, C_2, \cdots\}$ of cubes of this set which covers all of NA and is such that

[8] By a cube we mean a cube with sides parallel to the axes.

[9] Hans Rademacher, *Eineindeutige Abbildung und Messbarkeit*, Monatshefte für Mathematik und Physik, vol. 27 (1916), Theorem II, p. 190, with "circles centered at P" replaced by "squares centered at P." It is not necessary to use the theorem related to Vitali's covering theorem. One may instead consider a network of cubes; or one may give a proof like A. P. Morse's proof of his Theorem 4.3 (loc. cit., pp. 68–69). This remark applies also to our later applications of the theorem related to Vitali's covering theorem. The author is indebted to A. P. Morse for his having suggested the applicability of Vitali's covering theorem.

$$\sum_k |C_k| \leqq |N| + 1. \tag{4.4}$$

The sequence $\{f(NC_k)\}$ covers $f(NA)$. Let δ_k be the diameter of $f(NC_k)$. Then (4.3) and (4.4) imply that $\delta_k < \alpha$ ($k=1, 2, \cdots$) and

$$\sum_k \delta_k^m < \frac{\epsilon}{|N|+1} \sum_k |C_k| \leqq \epsilon.$$

Hence $f(NA)$ is an m-null set. This completes the first part of the proof.

We consider now the set B of critical points of rank r, where r is any positive integer less than m. To prove that $f(B)$ is an m-null set we shall show that there is a neighborhood N of each point of B such that $f(NB)$ is an m-null set.

Consider a point x_0 of B. Introduce the change of variables of §3 and consider the map (3.2, 3.3) in a closed cube K centered at u_0. Let D be the set of critical points of (3.2, 3.3) of rank r in K.

Suppose that u_1 is a point of D and u_2 is a point of K. Let $y_e = F(u_e)$ ($e=1, 2$). Then by the mean value theorem,

$$\begin{aligned} \overset{d}{y}_2 - \overset{d}{y}_1 &= \overset{d}{u}_2 - \overset{d}{u}_1, & d &= 1, \cdots, r, \\ \overset{r+h}{y}_2 - \overset{r+h}{y}_1 &= \sum_{c=1}^{r} F_c^{r+h}(u_1)(\overset{c}{u}_2 - \overset{c}{u}_1) + \sum_{i=1}^{m} \zeta_i^{r+h}(\overset{i}{u}_2 - \overset{i}{u}_1), & & \\ & & h &= 1, \cdots, n-r, \end{aligned} \tag{4.5}$$

where $\zeta_i^{r+h} \to 0$ as $u_2 \to u_1$.

Let $C(\gamma)$ be a closed cube of side γ in the space $(u^1, \cdots, u^m)$; and let $\Pi(\gamma)$ be the projection of $C(\gamma)$ on the space $(u^1, \cdots, u^r)$. Let p be a positive integer. Divide $\Pi(\gamma)$ by bisections into 2^{rp} congruent closed r-cubes each of side $\gamma/2^p$. Each such subcube is the projection of a *strip* of $C(\gamma)$ in which $u^1, \cdots, u^r$ may change by at most $\gamma/2^p$ and $u^{r+1}, \cdots, u^m$ may change by at most γ.

Consider any strip S that contains a point of D. Let u_1 and u_2 be any two points of S and let $y_e = F(u_e)$ ($e=1, 2$). Then (4.5) applied twice implies that

$$\begin{aligned} |\overset{d}{y}_2 - \overset{d}{y}_1| &\leqq \gamma/2^p, & d &= 1, \cdots, r, \\ |\overset{r+h}{y}_2 - \overset{r+h}{y}_1| &\leqq 2rU\gamma/2^p + 2m\zeta\gamma, & h &= 1, \cdots, n-r, \end{aligned} \tag{4.6}$$

where U is the least upper bound of the functions $|F_c^{r+h}(u)|$ for all u in K and ζ is the least upper bound of the functions $|\zeta_i^{r+h}|$ for all u_1 in D, u_2 in K such that $|u_2^i - u_1^i| \leqq \gamma$ ($c=1, \cdots, r$; $h=1, \cdots, n-r$; $i=1, \cdots, m$). Then $\zeta \to 0$ with γ.

Let δ be the diameter of $F(S)$. The triangle inequality and (4.6) imply that

$$\delta \leqq (V/2^p + W\zeta)\gamma, \tag{4.7}$$

where $V=r+(n-r)2rU$ and $W=(n-r)2m$. V and W are constants, and $V\geqq r\geqq 1$.

Let α and ϵ be given, $\alpha>0$, $1>\epsilon>0$. Put

$$\eta = \epsilon/(|K| + 1). \tag{4.8}$$

There exists a number G,

$$0 < G < \alpha/(V + 1), \tag{4.9}$$

such that $\gamma<G$ implies that

$$W\zeta < \eta^{1/(m-r)}(4V)^{-r/(m-r)}/2 < 1. \tag{4.10}$$

Let p be the positive integer determined by the relation

$$2V(4V)^{r/(m-r)}\eta^{-1/(m-r)} < 2^p \leqq 4V(4V)^{r/(m-r)}\eta^{-1/(m-r)}. \tag{4.11}$$

(Note that the first member of (4.11) is greater than 1, as it should be.)

Then $\gamma<G$ implies that $\delta<\alpha$ by (4.7), (4.9) and (4.10), since $2^p>1$; and $\gamma<G$ also implies that

$$\delta < \eta^{1/(m-r)}(4V)^{-r/(m-r)}\gamma \tag{4.12}$$

by (4.7), the first half of (4.11) and (4.10). Now consider all strips S containing points of D. There are at most 2^{rp} such strips; for these

$$\sum \delta^m < 2^{rp}\eta^{m/(m-r)}(4V)^{-mr/(m-r)}\gamma^m \leqq \eta\gamma^m = \eta\,|\,C(\gamma)\,|, \tag{4.13}$$

by (4.12) and the second half of (4.11). Thus $\gamma<G$ implies that there are sets covering $F[DC(\gamma)]$ each of whose diameters is less than α and the sum of the mth powers of whose diameters is less than $\eta|C(\gamma)|$.

Consider the set of all cubes $C(\gamma)$ centered at points of D and such that $\gamma<G$. As before, a sequence $\{C_k\}$ of these cubes covers D and is such that

$$\sum_k |C_k| \leqq |K| + 1. \tag{4.14}$$

Then the covering of $F(D)$ consisting of all the coverings of the sets $F(DC_k)$ is a covering by sets each of whose diameters is less than α and the sum of the mth powers of whose diameters is less than $\eta\sum_k|C_k| \leqq \epsilon$, by (4.13), (4.8) and (4.14). Hence $F(D)$ is an m-null set.

Let N be the inverse image under (3.1) of the interior of K. Then N

is a neighborhood of x_0. Further $f(NB)$ is contained in $F(D)$ and is therefore an m-null set. This completes the proof of Theorem 4.1.

5. **A. P. Morse's theorem.** We state the following theorem, due to A. P. Morse.[10]

THEOREM 5.1. *Given a positive integer q and a set A in the space of the variables x. There exists a sequence $A_0, A_1, A_2, \cdots$ of sets with the following properties*: (i) $A=\sum_{k=0}^{\infty} A_k$, (ii) A_0 *is denumerable*, (iii) A_k $(k=1, 2, \cdots)$ *is bounded*, (iv) *if $g(x)$ is any function of class C^q whose critical set includes A and if x_1 and x_2 are points of A_k, then*

$$\lim_{x_2\to x_1} \frac{g(x_2)-g(x_1)}{|x_2-x_1|^q}=0, \qquad k=1, 2, \cdots .$$

6. **Critical points of rank zero.** We prove Theorem 6.1.

THEOREM 6.1. *Let A be the set of critical points of rank zero of the map* (1.1). *Then $f(A)$ is an s-null set if $s \geqq m/q$.*

PROOF. Decompose A into the subsets of Theorem 6.1. Then $f(A_0)$ is denumerable and hence is an s-null set. We shall prove that $f(A_k)$ is an s-null set $(k=1, 2, \cdots)$. It will follow that $f(A)$ is an s-null set.

Consider a non-empty A_k (k fixed and positive). For simplicity put $B=A_k$. Given $\alpha>0$ and $\epsilon>0$. Given a point x of B. Let $C(\gamma)$ be the closed cube of side γ *centered at* x. By continuity and Theorem 5.1 there is a positive $G_x<1$ such that $\gamma<G_x$ implies that

$$\delta\{f[C(\gamma)]\}<\alpha \tag{6.2}$$

and

$$|f^j(x_1)-f^j(x)|<\frac{\epsilon^{1/s}2^{q-1}}{m^q n(|B|+1)^{1/s}}|x_1-x|^q, \quad j=1, \cdots, n, \tag{6.3}$$

whenever x_1 is in $BC(\gamma)$. Now by the triangle inequality $|x_1-x| \leqq m\gamma/2$ if x_1 is in $BC(\gamma)$. Hence (6.3) twice and the triangle inequality imply that

$$\delta\{f[BC(\gamma)]\}<2\frac{\epsilon^{1/s}2^{q-1}}{m^q(|B|+1)^{1/s}}\left(\frac{m\gamma}{2}\right)^q=\frac{\epsilon^{1s}/\gamma^q}{(|B|+1)^{1/s}},$$

and therefore

$$\delta\{f[BC(\gamma)]\}^s<\epsilon\gamma^{qs}/(|B|+1), \tag{6.4}$$

providing that $\gamma<G_x$.

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[10] Loc. cit., Theorem 4.2. By denumerable we mean denumerably infinite or finite. The sets A_k $(k=1, 2, \cdots)$ are condensed; however we do not make use of this property. The symbol $|x_2-x_1|$ denotes the distance between x_1 and x_2.

Consider the set of all cubes $C(\gamma)$, $\gamma < G_x$, centered at points x of B. As before, a sequence $\{C_k\}$ of these cubes covers B and is such that

$$\sum_k |C_k| \leqq |B| + 1. \tag{6.5}$$

Let γ_k be the side of C_k $(k=1, 2, \cdots)$.

The sequence $\{f(BC_k)\}$ covers $f(B)$. Further $\delta[f(BC_k)] < \alpha$ by (6.2); and

$$\sum_k \delta[f(BC_k)]^s < \epsilon \sum_k \gamma_k^{qs}/(|B| + 1) \leqq \epsilon \sum_k |C_k| /(|B| + 1) \leqq \epsilon$$

by (6.4) and (6.5), since $\gamma_k \leqq 1$ by construction, $qs \geqq m$ by hypothesis and $|C_k| = \gamma_k^m$ $(k=1, 2, \cdots)$. Hence $f(B)$ is an s-null set.

7. **Case II:** $m > n$. We prove Theorems 7.1 and 7.2.

THEOREM 7.1. *Let A be the set of critical points of rank r of the map* (1.1). *Then $f(A)$ is an n-null set if $q \geqq (m-r)/(n-r)$, $m > n$.*

PROOF. If $r=0$ the theorem reduces to Theorem 6.1 with $s=n$.

Suppose that $0 < r < n$. To prove that $f(A)$ is an n-null set we shall show that there is a neighborhood N of each point of A such that $f(NA)$ is an n-null set.

Consider a point x_0 of A. Introduce the change of variables of §3 and consider the map (3.2, 3.3) in a closed neighborhood $\overline{N}$ of u_0. Regard $u^1, \cdots, u^r$ as parameters for each permissible set of values of which (3.3) defines a map of the $(m-r)$-space $(u^{r+1}, \cdots, u^m)$ into the $(n-r)$-space $(y^{r+1}, \cdots, y^n)$. For each $(u^1, \cdots, u^r)$ let $\mathfrak{M}^*$ be the functional matrix of (3.3). Then $\mathfrak{M}'$ is of rank r if and only if $\mathfrak{M}^*$ is of rank zero.

Thus if $(u^1, \cdots, u^m)$ is a critical point of (3.2, 3.3) of rank r, then $(u^{r+1}, \cdots, u^m)$ is a critical point of (3.3) of rank zero for the values $u^1, \cdots, u^r$ of the parameters. But for each set of values of the parameters, the critical points of (3.3) of rank zero map into an $(n-r)$-null set, by Theorem 6.1 and hypothesis.

Let B be the set of critical points of (3.2, 3.3) of rank r in $\overline{N}$. The cross-section of $F(B)$ for each $(y^1, \cdots, y^r)$ is thus an $(n-r)$-null set. Further $F(B)$ is closed and therefore measurable. Hence $F(B)$ is an n-null set by the theorem of Fubini.

THEOREM 7.2. *If $m > n$ the critical values of the map* (1.1) *constitute an n-null set providing that $q \geqq m-n+1$.*

PROOF. Apply Theorem 7.1 with $r=0, 1, \cdots, n-1$. For these values of r, $m-n+1 \geqq (m-r)/(n-r)$.

8. **The hypothesis on q in Theorem 7.2 cannot be weakened.** Let $W(x^1, \cdots, x^t)$ be a function of t variables, $x^1, \cdots, x^t$, of class C^{t-1}, which takes on every value from 0 to 1 on a Jordan arc of critical points ($t \geqq 2$). H. Whitney has constructed such a function.[11] Consider the map

$$(8.1) \qquad y^1 = W(x^1, \cdots, x^{m-n+1}), \qquad y^a = x^{m-n+a}, \quad a = 2, 3, \cdots, n;\ m > n,$$[12]

in the unit cube. The map (8.1) is of class C^{m-n}, but the critical values of (8.1) do not constitute an n-null set. Indeed the set of critical values of (8.1) is the unit cube and is thus of n-dimensional measure 1.

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[11] Loc. cit.

[12] If $n=1$, (8.1) is to consist of the equation for y^1 only.