THE CAUCHY THEOREM FOR FUNCTIONS ON CLOSED SETS

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The object of this paper is to extend the theorem of Cauchy to functions of a complex variable defined on any bounded closed set, $E$, by determining conditions on $f(z)$ in order that for certain coverings of $E$, $C_n$, and an extension of $f(z)$, $f^*(z)$, $\lim_{n \to \infty} \int_{C_n} f^*(z) \, dz = 0$. It was suggested partly by the notion of a general monogenic function due to Trjitzinsky and partly by the measure theory methods of Menchoff and others, which succeed so well in lightening the restrictions on the real and imaginary parts of a complex function in order that $f(z)$ be regular.

Throughout this paper we shall consider only rectangles with sides parallel to the real and imaginary axes. A $C$-covering of a plane set $F$, denoted by $C$, will be a set of closed rectangles, possibly abutting, but nonoverlapping, which contain $F$. $c$ will denote the boundary of $C$. The covering $C_n$ is to be composed of rectangles $R_{mn}$ so that $C_n = \bigcup_{m,n} R_{mn}$, $(m, n = 1, 2, \ldots)$.

1. The extension, $f^*(z)$. If $u(P)$ is a positive continuous function defined on the closed and bounded set $F$ in the plane, we shall let $u^*(P) = \max_{Q \in F} u(Q) \left\{ 2 - d(P, Q)/d(P, F) \right\}$ for $P$ not in $F$, and $u^*(P) = u(P)$ for $P$ in $F$, where $d(P, Q)$ denotes the distance from $P$ to $Q$ and $d(P, F)$ the distance from the set $F$ to $P$. In general, if $u(P)$ is continuous, since $u(P) = (u(P) + (u(P))/2 - (|u(P)| - u(P))/2)$, that is, since $u(P)$ is the difference of two continuous positive functions, $u^*(P)$ will denote the extension of $u(P)$ obtained by extending as before these parts. If $f(z) = u(x, y) + iv(x, y)$ is defined on a bounded closed set and continuous, $f^*(z)$ will denote $u^*(x, y) + iv^*(x, y)$.

LEMMA 1. If $u(P)$ is defined on a bounded closed set $F$ and $|u(Q) - u(P)| < M(P)d(P, Q)$ where $M(P)$ is a finite function of $P$ defined on $F$, then $|u^*(P) - u^*(Q)| < 20 M(P) d(P, Q)$, for $P$ in $F$ and $Q$ arbitrary.

Presented to the Society, December 29, 1941; received by the editors February 25, 1942.

1 W. J. Trjitzinsky, Théorie des Fonctions d'une Variable Complexe Définies sur des Ensembles Généraux, Annales Scientifique de L'École Normale Supérieure, Paris, 1938, p. 120.


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PROOF. Consider first the case for which \( u(P) \geq 0 \). If \( Q \) is any point not in \( D \), if \( Q_0 \) is a point in \( F \) for which \( d(Q, Q_0) = d(Q, F) \), and if, of the points \( S \) satisfying the inequality \( d(S, Q) \leq 2d(Q, Q_0) \), \( R \) is the point where the maximum of \( u \) is attained, then \( u^*(Q_0) \leq u^*(Q) \leq u^*(R) \). Hence
\[
|u^*(Q) - u^*(P)| \leq |u^*(Q) - u^*(R)| + |u^*(R) - u^*(P)|
\]
\[
\leq |u^*(Q_0) - u^*(P)| + 2|u^*(R) - u^*(P)|
\]
\[
< M(P)d(Q_0, P) + 2M(P)d(R, P).
\]
It is easily verified that \( d(Q_0, P) \leq 2d(Q, P) \) and \( d(R, P) \leq 4d(Q, P) \), so that \( |u^*(Q) - u^*(P)| < 10M(P) d(P, Q) \) and the lemma is proved for case of \( u(P) \) positive. In the general case, \( u(P) = (|u| + u)/2 - (|u| - u)/2 = g(P) - h(P), \) where \( g \) and \( h \) are positive functions, and satisfy the conditions of the lemma, so that for \( P \) in \( F \) and \( Q \) arbitrary \( |g^*(Q) - g^*(P)| \) and \( |h^*(Q) - h^*(P)| \) are each less than \( 10M(P) d(P, Q) \). Hence for \( u^* = g^* - h^* \), it readily follows that \( |u^*(Q) - u^*(P)| < 20M(P) d(P, Q) \), and the proof of the lemma is complete.

2. Bounded derivatives. We shall use the following fundamental lemma:

**Lemma 2.** Let \( w(x, y) \) be a real, continuous function defined in the square \( S \), the sides of which are parallel to the coordinate axes, and let \( F \) be a closed set in \( S \) and such that
\[
|w(x + h, y) - w(x, y)| \leq M|h|,
\]
\[
|w(x, y + k) - w(x, y)| \leq M|k|
\]
for all points \( (x, y) \) in \( F \) and for all points \( (x + h, y), (x, y + k) \) of the square \( S \), where \( M \) is a constant. Finally let \( R \) be the least rectangle with sides parallel to the axes containing \( F \).

Under these conditions the following inequalities hold:
\[
\left| \int_{x_1}^{x_2} [w(x, y) - w(x, y_1)] dx - \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{\partial w}{\partial y} \, dx \, dy \right| \leq 5Mm(S - F),
\]
\[
\left| \int_{y_1}^{y_2} [w(x, y) - w(x, y_1)] dy - \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial w}{\partial x} \, dx \, dy \right| \leq 5Mm(S - F)
\]
where \( (x_1, y_1), (x_2, y_1), (x_2, y_2) \) and \( (x_1, y_2), (x_1 \leq x_2, y_1 \leq y_2) \) are the

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4 For the proof of this lemma, cf. loc. cit., p. 10.
5 The "least rectangle" may be only a segment or a point.
6 The conditions on \( w \) imply the existence of the partial derivatives a.e. in \( F \).
corners of the rectangle \( R \) and \( m(S - F) \) is the measure of the set \( S - F \), which is composed of points of \( S \) not in \( F \).

**Theorem 1.** Let \( f(z) (= u(x, y) + iv(x, y)) \) be defined on the bounded closed set \( E \), and let \( R \) be a rectangle with sides parallel to the axes containing at least one point of \( E \) on each side. If (letting \( F = E \cdot R \))

(1) for all \( z \) and \( z + h \) in \( F \), and a constant \( B \),

\[
\left| \frac{f(z + h) - f(z)}{h} \right| < B,
\]

(2) the Cauchy-Riemann equations hold a.e. (almost everywhere) in \( F \), where the partial derivatives of \( u \) and \( v \) exist, then \[ \int f^*(z)dz \] \( < 400Bm(S - F) \) where \( r \) is the boundary of \( R \), and \( S \) is a square of least area containing \( R \).

**Proof.** If \( h = k + il \), condition (1) implies

\[
\left| \frac{u(x + k, y + l) - u(x, y)}{h} \right| < B,
\]

and a similar condition on \( v(x, y) \), for every point \( z \), and \( z + h \), in \( F \). According to Lemma 1,

\[
\left| \frac{u^*(x + k, y + l) - u^*(x, y)}{h} \right| < 20B,
\]

for each point \( z \) in \( F \), and \( z + h \) in \( R \). Hence by Lemma 2,

\[
\left| \int_{x_1}^{x_2} [u^*(x, y_2) - u^*(x, y_1)]dx - \int_{y_1}^{y_2} \frac{\partial u^*}{\partial y} dy \right| < 100Bm(S - F),
\]

(\( x_1, y_1 \)) and (\( x_2, y_2 \)) being corners of \( R \). Similar inequalities for \( u^*(x, y) \) with respect to \( y \), and \( v^*(x, y) \) with respect to \( x \) and \( y \) also hold. But

\[
\int f^*(z)dz = \int u^*dx - v^*dy + i \int v^*dx + u^*dy
\]

and

\[
\int u^*dx = -\int_{x_1}^{x_2} [u^*(x, y_2) - u^*(x, y_1)]dx.
\]

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This is in no way a further restriction on \( E \), for almost all points of any measurable plane set are points of linear density for it both in the direction of the \( x \)-axis and in that of the \( y \)-axis.

The condition that the limits, \( \lim_{h \to 0} (f(z+h) - f(z))/h \), as \( z + h \) approaches \( z \) through points of \( E \) along either of two curves having non-collinear tangents at \( z \), should be equal, is equivalent to the condition (2) in the presence of (1). For (1) and (2) imply that \( f^*(z) \) is monogenic a.e. in \( E \) (Menchoff, loc. cit., Theorem 2, p. 27 and Theorem 5, p. 23).
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Let \( \epsilon' = \int_{\mathbb{R}} \left[ u^*(x, y_2) - u^*(x, y_1) \right] \, dx - \int_{\mathbb{R}} \left( \frac{\partial u^*}{\partial y} \right) dy \, dx \). Then \( \int u^* \, dx = -\int \left( \frac{\partial u^*}{\partial y} \right) dy \, dx - \epsilon' \). Taking into account similar reasoning for the other parts of \( \int f^*(z) \, dz \) we have

\[
\int f^*(z) \, dz = -\int \int F \left( \frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x} \right) \, dxdy + i \int \int F \left( \frac{\partial u^*}{\partial x} - \frac{\partial v^*}{\partial y} \right) \, dxdy + \epsilon
\]

where \( |\epsilon| < 400Bm(S - F) \). Since the Cauchy-Riemann equations hold a.e. in \( F \), \( \int f^*(z) \, dz = \epsilon \), and the proof of the theorem is complete.

**Corollary.** Let \( f(z) \) be defined on the bounded closed set \( E \) with a bounded derivative there. Let \( S = \sum S_m \) be a \( C \)-covering of \( E \) by squares with \( R_m \) the least rectangle within \( S_m \)-\( E \). Then there is a constant \( B \) for which \( \sum S_m |\int_{R_m} f^*(z) \, dz| < 400Bm(S - E) \), and if \( S \rightarrow E \), \( C = \sum R_m \rightarrow E \), and \( \lim_{C \rightarrow E} \int f^*(z) \, dz = 0 \).

3. Derivatives finite, except for a denumerable set. We prove this theorem:

**Theorem 2.** If \( f(z) \) is defined and continuous on the bounded closed set \( E \), and if, except for a denumerable number of points, \( \lim_{h \rightarrow 0} \frac{|f(z+h) - f(z)|}{h} < \infty \), and the Cauchy-Riemann equations hold a.e. where the partial derivatives of \( u \) and \( v \) exist, then there is a sequence of \( C \)-coverings, \( \{ C_n \} \), for which \( \lim_{n \rightarrow \infty} \sum |\int_{C_n} f^*(z) \, dz| = 0 \).

**Proof.** Define \( I(C) = \sum |\int_{C} f^*(z) \, dz| \). If for every point \( z \) of \( E \) there is a neighborhood \( N(z) \) such that for every closed subset of \( E \) in \( N \), there is a sequence of coverings \( \{ C_n \} \) for which \( \lim_{n \rightarrow \infty} I(C_n) = 0 \), then by the Heine-Borel theorem there exists a sequence of coverings of \( E \) with the property mentioned in the theorem. The proof will be complete therefore, if we show that there is such a neighborhood for each point of \( E \). Let \( P \) be those points of \( E \) such that in every neighborhood of \( z \) there is a closed subset of \( E \) for which there is no sequence of \( C \)-coverings, \( \{ C_n \} \), for which \( \lim_{n \rightarrow \infty} I(C_n) = 0 \). We shall assume that \( P \) is not empty and show that this leads to an absurdity.

Let \( P_m \) \((m = 1, 2, \cdots)\) be the points of \( P \) for which each of the absolute values,

\[
|u^*(x + k, y) - u^*(x, y)|, \quad |v^*(x + k, y) - v^*(x, y)|, \\
|u^*(x, y + k) - u^*(x, y)|, \quad |v^*(x, y + k) - v^*(x, y)|
\]

is less than or equal to \( m\,|k| \) for \( |k| \leq 1/m \), \( k \) a real number. Since
and \( v^* \) are continuous and \( P \) is closed, \( P_m \) is closed. Since at each point of \( E \), except for a denumerable set \( H \), the partial derivatives are finite, \( P = \sum_m P_m + P \cdot H \). By Baire's theorem, there is an isolated point of \( P \) in \( H \), or for some integer \( N \) there is a point \( z_0 \) in \( P \), the center of a square \( S \) which contains only points of \( P \) which are in \( P_N \). The former alternative is quickly dismissed as impossible; we proceed on the basis of the latter, and let \( F \) be any closed subset of \( E \cdot S \). Subdivide the sides of \( S \) into \( n \) equal parts, \( n > 2N \), and obtain the squares \( S_j \) \((j = 1, 2, \ldots, n^2)\). \( \epsilon \) being given, choose \( n \) so great that the squares \( \overline{S}_j \) which contain points of \( F \cdot P \) satisfy the inequality, \( m(\sum_j \overline{S}_j - P \cdot S) < \epsilon/800N \). If \( \overline{R}_j \) is the least rectangle containing \( P \cdot \overline{S}_j \), and \( \overline{C} \) is the covering \( \sum_j \overline{R}_j \), by Theorem 1, \( I(\overline{C}) < 400N \sum_j m(\overline{S}_j - P \cdot \overline{S}_j) < \epsilon/2 \). Since \( I(R) \) is a continuous function of \( r \), \( C \) may be extended by the addition of more small rectangles, so that, if \( C' \) is the new covering, \( I(C') \) remains less than \( \epsilon/2 \), but so that the points of \( F \cdot P \) are inner points of the covering. The part of \( F \) not already covered (denote it by \( G \)) is such that its closure contains only points \( z \) of \( F \) for which there is some neighborhood \( N(z) \) with the property that every closed subset of \( F \) in \( N \) can be \( C \)-covered, say by \( C_n(z) \) \((n = 1, 2, \ldots)\) and \( \lim_{n \to \infty} I(C_n(z)) = 0 \). Let \( S(z) \) be a square with \( z \) as center entirely within \( N(z) \). Of these squares a finite number, \( k \), cover \( G \), and within each of these is a covering, \( C(z) \), of \( G \) for which \( I(C(z)) < \epsilon/2k \). Hence \( G \) is \( C \)-covered by a covering \( C \) for which \( I(C) < \epsilon/2 \). \( F \) is therefore \( C \)-covered by \( C + C' \) for which \( I(C + C') < \epsilon \), so that \( z_0 \) cannot belong to \( P \), contrary to assumption. This completes the proof of Theorem 2.

**Corollary.** If \( f(z) \), defined on the bounded closed set \( E \) and continuous there, has a derivative at each point except at most a denumerable set, there is a sequence of \( C \)-coverings of \( E \) with \( E \) as their limit for which \( \lim_{n \to \infty} \int_{C_n} f^*(z) dz = 0 \).

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