THE ZEROS OF CERTAIN COMPOSITE POLYNOMIALS

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1. Introduction. If \( A_0(z) \) is a given \( m \)th degree polynomial and

\[
A_k(z) = (\beta_k - z)A'_{k-1}(z) + (\gamma_k - k)A_{k-1}(z),
\]

where \( \gamma_k \neq m + k \),

\[ k = 1, 2, \ldots, n, \]

we may obtain various theorems on the relative location of the zeros of \( A_0(z) \) and \( A_n(z) \) by the familiar method of first finding such relations for two successive \( A_k(z) \) and then iterating the relations \( n \) times.

This method has already been employed in the study of the zeros of sequence (1.1) for the following three cases: (1) for all \( k \), \( \beta_k = 0 \) and \( \gamma_k \) is real;\(^1\) (2) for all \( k \), \( \gamma_k = m + 1 \)—a limiting case leading to Grace’s theorem;\(^2\) and (3) the limiting case that for all \( k \), as \( h \to 0 \), \( h\beta_k \to \beta'_k \) and \( h(\gamma_k - k) \to 1 \), in which case \( \lim h^k A_k(z) \) is a linear combination of \( A_0(z) \) and its first \( k \) derivatives.\(^3\)

In the present article we propose to apply the method to the case that the parameters \( \beta_k \) and \( \gamma_k \) are complex numbers represented by points within certain given regions of the plane.

To calculate the \( n \)th iterate \( A_n(z) \) in our case, let us define

\[
A(z) \equiv A_0(z) \equiv a_0 + a_1z + \cdots + a_mz^m;
\]

\[
B(z) \equiv (\beta_1 - z)(\beta_2 - z) \cdots (\beta_n - z)
\]

\[ = b_0 + b_1z + \cdots + b_nz^n, \]

\[
C(z) \equiv (\gamma_1 - 1 - z)(\gamma_2 - 2 - z) \cdots (\gamma_n - n - z);
\]

\[
S(z, k, \rho) \equiv B(z) \sum \frac{\gamma_1^{(k+p)}}{\beta_1 - z} \frac{\gamma_2^{(k+p)}}{\beta_2 - z} \cdots \frac{\gamma_{n-p}^{(k+p)}}{\beta_{n-p} - z} - (n - \rho),
\]

where \( \gamma_j^{(r)} \equiv \gamma_j - r \) thus \( \gamma_j^{(r)} - j \) is a zero of \( C(z+r), \rho < n \), and the sum is formed for all \( j_i \) such that \( 1 \leq j_1 < j_2 < \cdots < j_{n-p} \leq n; \)

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\[ S(z, k, n) = B(z) \quad \text{and} \quad S(z, k, p) = 0 \quad \text{for} \quad p > n. \]

Then by repeated use of formula (1.1), we find for
\[ D(z) \equiv A_n(z) = d_0 + d_1z + \cdots + d_mz^m \]
the two expressions
\[ D(z) = \sum_{p=0}^{n} S(z, 0, p) \frac{d^p A(z)}{dz^p} \]
and
\[ D(z) = \sum_{k=0}^{m} \sum_{p=0}^{m-k} \frac{(k + p)!}{k!} S(0, k, p) a_{k+p}z^k. \]

Let us note two special cases of these formulas. First, if \( \beta_k = 0 \) for all \( k \), then
\[ S(0, k, p) = 0 \quad \text{for} \quad p \neq 0, \quad S(0, k, 0) = C(k) \]
and, hence,
\[ D(z) = C(0)a_0 + C(1)a_1z + \cdots + C(m)a_mz^m. \]
Secondly, if, for all \( k \), \( \gamma_k = \gamma + 1 \), where \( \gamma \) is any constant other than \( m, m+1, \cdots, m+n-1 \), then
\[ S(0, k, p) = (\gamma - k - p)(\gamma - k - p - 1) \cdots (\gamma + 1 - k - n) \sum \beta_i \beta_{i+1} \cdots \beta_j \]
\[ = (-1)^{n-p}(n-p)! C_{\gamma-k-p,n-p} b_{n-p} \]
where \( C_{r,s} = r(r-1) \cdots (r-s+1)/1 \cdot 2 \cdot \cdots s \) and, hence, except for the multiplier \( n! \),
\[ D(z) = \sum_{k=0}^{m} \sum_{p=0}^{m-k} (-1)^{n-p} C_{n,p}^{-1} C_{\gamma-k-p,n-p} C_{b_{k+p},b_{n-p}}z^k \]
with \( b_{n-p} = 0 \) for \( p > n \).

In what follows it will be convenient to denote by a script capital \( \mathcal{F} \) a region containing all the zeros of a given function \( F(z) \). Thus, \( \mathcal{A} : |z| \leq r \) will mean that all the zeros of the polynomial \( A(z) \) lie in or on the circle \( |z| = r \).

2. Zeros of two successive \( A_k(z) \). Using the preceding notation, the following lemma may be stated.

**Lemma.** Let \( \gamma_j = \gamma_j - j \) denote the zeros of \( C(z) \). Then,
(a) \( \mathcal{A}_k : r_1 \leq |z| \leq r_2 \) and \( |\beta_k| \leq \lambda r_1 \) imply
\[ A_{k+1}: r_1 \min \left[ 1, \frac{\gamma_k \mid \alpha - m\lambda}{\gamma_k' - m} \right] \leq |z| \]

(2.1)

\[ \leq r_2 \max \left[ 1, \frac{|\gamma_k'| + m\lambda}{|\gamma_k' - m|} \right]; \]

(b) \( A_k: |z| \leq r \) and \( |\beta_k| \geq \lambda r \) imply

(2.2)

\[ A_{k+1}: |z| \leq r \text{ and } |z| \geq r \max \left[ 1, \frac{m\lambda - |\gamma_k'|}{|m - \gamma_k'|} \right]; \]

(c) \( A_k: \omega_1 \leq \arg z \leq \omega_2 \) with \( \omega_2 - \omega_1 \leq \pi \) and \( \beta_k = 0 \) imply

(2.3)

\[ A_{k+1}: \omega_1 + \min \left( 0, \arg \frac{\gamma_k'}{\gamma_k' - m} \right) \leq \arg z \]

\[ \leq \omega_2 + \max \left( 0, \arg \frac{\gamma_k}{\gamma_k' - m} \right). \]

This lemma may be deduced from the results of a previous paper\(^4\) or may be proved directly as follows.

Let \( \mathcal{A} \) be a circular region and let \( \zeta \) be any zero of \( A_{k+1}(z) \) outside \( \mathcal{A} \). Then, by Laguerre's theorem,\(^5\) there exists a point \( \alpha \) in \( \mathcal{A} \) such that \( \left| A_k'(\zeta)/A_k(\zeta) \right| = m/(\zeta - \alpha) \) and, hence, by (1.1)

(2.4)

\[ \zeta = \frac{\gamma_k' \alpha - m\beta_k}{\gamma_k' - m}. \]

In particular for \( |\beta_k| \leq \lambda r_1 \), if \( \mathcal{A}_k: |z| \leq r_2 \), then\(^6\) we have that

\[ |\zeta| \leq r_2(\gamma_k' + m\lambda) |\gamma_k' - m|^{-1} \]

whereas if \( \mathcal{A}_k: |z| \geq r_1 \), then

\[ |\zeta| \geq r_1(\gamma_k' - m\lambda) |\gamma_k' - m|^{-1}. \]

Hence, if all the zeros of \( A_k(z) \) lie in the ring \( r_1 \leq |z| \leq r_2 \), an arbitrarily chosen zero of \( A_{k+1}(z) \) must lie in the ring (2.1).

If \( |\beta_k| \geq \lambda r \) and \( \mathcal{A}_k: |z| \leq r \), then

\[ |\zeta| \leq r(m\lambda - |\gamma_k'|) |\gamma_k' - m|^{-1} \]

and hence the zeros of \( A_{k+1}(z) \) not satisfying the first inequality (2.2) must satisfy the second inequality (2.2).

Finally, for \( \beta_k = 0 \), if \( \mathcal{A}_k: \omega \leq \arg z \leq \omega + \pi \), then \( \omega + \arg [\gamma_k' (\gamma_k' - m)^{-1}] \leq \arg \zeta \leq \omega + \pi + \arg [\gamma_k' (\gamma_k' - m)^{-1}] \). Setting \( \omega = \omega_1 \) and \( \omega = \omega_2 - \pi \) and combining the results, we conclude that, if all the zeros of \( A_k(z) \) lie in the sector \( \omega_1 \leq \arg z \leq \omega_2 \), then all the zeros of \( A_{k+1}(z) \) lie in the sector (2.3).


\(^5\) Laguerre, Oeuvres, vol. 1 p. 49.

\(^6\) See M. Marden, ibid. p. 402.
3. Zeros of $A_0(z)$ and $A_n(z)$. We shall now apply part (1) of the lemma to the successive $A_k(z)$ in order to determine the relative location of the zeros of the polynomials $A(z)\equiv A_0(z)$, $B(z)$, $C(z)$ and $D(z)\equiv A_n(z)$. In addition to the notation used hitherto, we shall use the symbol $\mathfrak{C}(z)$ for the polynomial whose zeros are the moduli of the zeros of $C(z)$: 

$$\mathfrak{C}(z) = (\gamma_1 | z - z)(\gamma_2 | z - z) \cdots (\gamma_n | z - z).$$

**Theorem I.** Given the positive constants $\rho$ and $\lambda$ ($\lambda < 1$). Then,

1. $A: |z| \leq r$, $B: |z| \leq \lambda r$ and $C: \rho |z - m| \geq |z| + m\lambda$ imply $D: |z| \leq r \text{ max } (1, \rho^n);$ 
2. $A: |z| \leq r$, $B: |z| \leq \lambda r$ and $C: 0 < \rho |z - m| \leq |z| + m\lambda$ with $\rho \geq 1$ imply $D: |z| \leq r |\mathfrak{C}(-m\lambda)/C(m)|$; 
3. $A: |z| \geq r$, $B: |z| \leq \lambda r$ and $C: \rho |z - m| \geq |z| - m\lambda > 0$ with $\rho \leq 1$ imply $D: |z| \geq r |\mathfrak{C}(m\lambda)/C(m)|$; 
4. $A: |z| \geq r$, $B: |z| \leq \lambda r \text{ min } (1, \rho^n)$ and $C: 0 < \rho |z - m| \leq |z| - m\lambda$ imply $D: |z| \geq r \text{ min } (1, \rho^n)$.

To prove this theorem, let us define

$$\mu_k = |m - \gamma_k|^{-1}(|\gamma_k| + m\lambda);$$

$$M_k = \max \mu_1 \mu_2 \cdots \mu_k,$$  
where $\sigma_i = 0, 1$; 

$$v_k = |m - \gamma_k|^{-1}(|\gamma_k| - m\lambda) \text{ if } |\gamma_k| > m\lambda \text{ and }$$

$$v_k = 0 \text{ if } |\gamma_k| \leq \lambda m;$$

$$N_k = \min v_1 v_2 \cdots v_k,$$  
where $\sigma_i = 0, 1$.

If $A: |z| \leq r$ and $B: |z| \leq \lambda r$, then by the right side of (2.1)

$$A_1: |z| \leq r M_1, A_2: |z| \leq r M_2, \cdots, A_n: |z| \leq r M_n.$$ 

Since in part (1) of Theorem I

$$\mu_k \leq \rho,$$ 

$$M_n = \max (1, \rho^n),$$  
and, since in part (2) $\mu_k \geq 1$,

$$M_n = \mu_1 \mu_2 \cdots \mu_n = |\mathfrak{C}(-m\lambda)/C(m)|.$$ 

If $A: |z| \geq r$ and $B: |z| \leq \lambda r N_n$, then by the left side of (2.1)

$$A_1: |z| \geq r N_1, A_2: |z| \geq r N_2, \cdots, A_n: |z| \geq r N_n.$$ 

Since in part (3) of Theorem I $0 < v_k \leq \rho \leq 1$, $N_n = v_1 v_2 \cdots v_n = |\mathfrak{C}(m\lambda)/C(m)|$; whereas since in part (4) $v_k \geq \rho$, $N_n = \min (1, \rho^n)$. We have thus established Theorem I.
It is to be noticed that each region \( C \) of Theorem I is bounded by one of the ovals \( \rho |m - z| = |z| \pm m/ \) the cartesian curve

\[
[(\rho^2 - 1)(x^2 + y^2) - 2m^2x + m^2(\rho^2 - \lambda^2)] = 4m^2\lambda^2(x^2 + y^2)
\]

having ordinary foci at the three points \( z = 0, z = m \) and \( z = m(\rho^2 - 1)(\rho^2 - \lambda^2) \) and a singular focus at the point \( z = m\rho^2(\rho^2 - 1)^{-1} \).

If \( \rho > 1 \) curve (3.1) consists of two nested ovals both enclosing \( z = m \) and both excluding \( z = 0 \); in this case, the region \( C \) of part (1) of the theorem is the exterior of the outer oval, \( C \) of part (2) is the interior of the outer oval exclusive of point \( z = m \) and \( C \) of part (4) is the interior of the inner oval exclusive of point \( z = m \). If \( \rho = 1 \), curve (3.1) degenerates into the hyperbola with foci at \( z = 0 \) and \( z = m \) and transverse axis of \( m\lambda \); in this case \( C \) of part (1) is the region left of the left branch of the hyperbola, \( C \) of part (2) is the region right of the left branch not including \( z = m \), \( C \) of part (3) is the region common to the exterior of circle \( |z| = m\lambda \) and the left of the right branch and \( C \) of part (4) is the interior of the right branch with point \( z = m \) omitted.

If \( \lambda < \rho < 1 \), curve (3.1) consists of nested ovals, now however both containing \( z = 0 \) and excluding \( z = m \); in this case, \( C \) of part (1) is the interior of the inner oval, \( C \) of part (3) is the region common to the exterior of circle \( |z| = m\lambda \) and the interior of the outer oval and \( C \) of part (4) is the exterior of the outer oval exclusive of point \( z = m \). In the latter case, if \( \rho \to \lambda \), the inner oval shrinks to a point and hence, for \( \rho < \lambda \), \( C \) of part (1) is a null-set, and the \( C \)'s of parts (3) and (4) are those described for \( \lambda < \rho < 1 \).

In the foregoing discussion, we have implied that \( \lambda \neq 0 \). If \( \lambda = 0 \), curve (3.1) degenerates into the dipolar circle \( \rho |z - m| = |z| \) and \( D(z) \) is given by formula (1.7). We may thus state the following corollary.

**Corollary.** If all the zeros of a polynomial \( A(z) = a_0 + a_1z + \cdots + a_mz^m \) lie in the ring \( 0 < r_1 < |z| < r_2 \leq \infty \) and if all the zeros of an nth degree polynomial \( C(z) \) lie in the connected region bounded by the circles \( |z| = \rho_1|z - m| \) and \( |z| = \rho_2|z - m| \) with \( \rho_1 \leq \rho_2 \), then all the zeros of the polynomial \( D(z) = C(0)a_0 + C(1)a_1z + \cdots + C(m)a_mz^m \) lie in the ring

\[
r_1 \min (1, \rho_1^n) \leq |z| \leq \rho_2 \max (1, \rho_2^n).
\]

If \( \rho_2 < 1 \), the left side of (3.2) may be replaced by the then larger number

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8 For the cases (1) \( r_1 = 0, \rho_1 = 0, r_2 = 1 \); (2) \( r_2 = \infty, \rho_1 = 1, \rho_2 = \infty \); and (3) \( r_1 = r_2, \rho_1 = \rho_2 = 1 \), see N. Obrechkoff, *Sur les zeros des polynomials*, C. R. Acad. Sci. Paris vol. 209 (1939) pp. 1270–1272, and L. Weisner, *Roots of certain classes of polynomials*, Bull. Amer. Math. Soc. vol. 48 (1942) p. 283–286.
and, if \( 1 < \rho_1 \), the right side may be replaced by the then smaller number \( r_2 \left| \frac{C(0)}{C(m)} \right| \).

So far we have applied part (1) of the lemma to the successive \( A_k(z) \). Similarly, if we apply part (3) of the lemma and formula (1.7), we may obtain the following result.

**Theorem II.** If all the zeros of the polynomial \( A(z) = a_0 + a_1 z + \cdots + a_m z^m \) are in the sector \( \omega_1 \leq \arg z \leq \omega_2 \) with \( \omega_2 - \omega_1 = \omega \leq \pi \), and if all the zeros of an \( n \)th degree polynomial \( C(z) \) are in the lune \( \theta_1 \leq \arg \left[ \frac{z}{(z-m)} \right] \leq \theta_2 \) with \( |\theta_1| + |\theta_2| \leq (\pi - \omega) / n \), then all the zeros of the polynomial \( D(z) = a_0 C(0) + a_1 C(1) z + \cdots + a_m C(m) z^m \) lie in the sector

\[
\omega_1 + \min (0, n\theta_1) \leq \arg z \leq \omega_2 + \max (0, n\theta_2).
\]

If \( \theta_2 < 0 \), \( \min (0, n\theta_1) \) may be replaced in (3.3) by the then larger number \( \arg \frac{C(0)}{C(m)} \) and, if \( 0 < \theta_1 \), \( \max (0, n\theta_2) \) may be replaced by the then smaller number \( \arg \frac{C(0)}{C(m)} \).

4. **Entire functions.** Theorem II and the corollary to Theorem I may be generalized at once through replacing

\[
D(z) = a_0 C(0) + a_1 C(1) z + \cdots + a_m C(m) z^m = \delta (\delta_1 - z) (\delta_2 - z) \cdots (\delta_m - z)
\]

by

\[
F(z) = a_0 E(0) + a_1 E(1) z + \cdots + a_m E(m) z^m,
\]

where

\[
E(z) = e^{\lambda z} C(z) \quad \text{and} \quad \lambda = \mu + iv.
\]

In fact, since

\[
F(z) = \sum_{k=0}^{m} a_k C(k) e^{\lambda k z^k} = D(e^{\lambda z}) = \delta e^{\lambda \delta \prod_{k=1}^{m} (\delta_k e^{-\lambda} - z)},
\]

the substitution of \( E(z) \) and \( F(z) \) for \( C(z) \) and \( D(z) \) would require only the following changes: in the corollary to Theorem I, inequality (3.2) becomes

\[
e^{-\mu r_1} \min (1, \rho_1^n) \leq |z| \leq e^{-\mu r_2} \max (1, \rho_2^n)
\]

where \( e^{\mu n} \left| E(0) / E(m) \right| \) may replace \( \min (1, \rho_1^n) \) if \( \rho_2 \leq 1 \) and \( \max (1, \rho_2^n) \) if \( \rho_1 \leq 1 \); in Theorem II, inequality (3.3) becomes

\[
\omega_1 - \nu + \min (0, n\theta_1) \leq \arg z \leq \omega_2 - \nu + \max (0, n\theta_2)
\]
CERTAIN COMPOSITE FUNCTIONS

where \([mv+\arg E(0)/E(m)]\) may replace \(\min (0, \pi \theta_1)\) if \(\theta_2 \leq 0\) and \(\max (0, \pi \theta_2)\) if \(0 < \theta_1\).

Furthermore, these results may be extended to entire functions \(E(z)\) of genus zero or one provided the zeros of \(E(z)\) are assumed to lie in infinite regions, determined by taking \(\rho_1 = 1\) or \(\rho_2 = 1\) and \(\theta_1 = \theta_2 = 0\).

**Theorem III.** Given the entire functions

\[
A(z) = \sum_{k=0}^{m} a_k z^k, \quad E(z) = e^{\lambda z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\gamma_k}\right) e^{\lambda_k z},
\]

\[
F(z) = \sum_{k=0}^{m} a_k E(k) z^k,
\]

where \(\lambda_j = \mu_j + iv_j\).

(a) If all the zeros of \(A(z)\) lie in the ring \(0 \leq r_1 \leq |z| \leq r_2 \leq \infty\), if all the zeros of \(E(z)\) lie in the region \(\rho_1 \leq |z/(z-m)| \leq \rho_2\) with at least one number \(\rho_1, \rho_2\) unity, and if \(\mu_0 + \mu_1 + \cdots + \mu,\) then all the zeros of \(F(z)\) lie in the ring \(K_1 e^{-\nu r_1} \leq |z| \leq K_2 e^{-\nu r_2}\), where \(K_1 = 1\) or \(e^{\nu m} |E(0)/E(m)|\) according as \(\rho_1 = 1\) or \(\rho_1 < 1\) and \(K_2 = 1\) or \(e^{\nu m} |E(0)/E(m)|\) according as \(1 = \rho_2\) or \(1 < \rho_2\).

(b) If all the zeros of \(A(z)\) lie in the sector \(\omega_1 \leq \arg z \leq \omega_2\) with \(\omega_2 - \omega_1 \leq \pi\), if all the zeros of \(E(z)\) lie on the real axis outside of the segment \((0, m)\) and if \(\nu_0 + \nu_1 + \cdots + \nu,\) then all the zeros of \(F(z)\) lie in the sector \(\omega_1 - \nu \leq \arg z \leq \omega_2 - \nu\).

Theorem III(b) is a partial generalization of results due to Laguerre and Polya\(^1\) in the case that both \(\nu = 0\) and all the zeros of \(A(z)\) are real. However, it may also be derived from this special case by use of the theorem quoted in problem 153, p. 65, vol. 2 Polya-Szegö’s *Aufgaben der Analysis.* For this fact and its following proof, the author is indebted to the referee, Professor Polya.

We may assume without loss of generality that \(\nu = 0\). Then, according to the Laguerre-Polya results, \(\alpha_k = E(k)\) form a set of multipliers such that, if any polynomial \(A(z) = a_0 + a_1 z + \cdots + a_m z^m\) has only positive (negative) zeros, so has also the polynomial \(C(z) = a_0 a_1 + a_0 a_2 z + \cdots + a_m a_m z^m\). But such multipliers have also the property that, if all the zeros of \(A(z)\) lie in the sector \(\omega_1 \leq \arg z \leq \omega_2\) with \(\omega_2 - \omega_1 \leq \pi\), all the zeros of \(C(z)\) also lie in this sector. For, since all the zeros of \((1+z)^m\) are negative, the zeros of polynomial

\[
G(z) = a_0 + C_{m,1} \alpha_1 z + C_{m,2} \alpha_2 z^2 + \cdots + a_m \alpha_m z^m
\]

are also all negative, and, since the sector is a convex region contain-
ing the origin, the theorem from Polya-Szegö may be applied with the $F(z)$ of the theorem taken as $A(z)$. Theorem III(b) then follows immediately.

As an application of Theorem III, let us consider the polynomial $F(z) = \sum_{k=0}^{n} a_k G(k + \rho) z^k$ where $\rho > 0$ and $G(z) = \Gamma(z)^{-1} = e^\psi \prod_{n=1}^{\infty} (1 + n^{-1}z)e^{-z/n}$, the reciprocal of the gamma function. Since $\nu = 0$ and all the zeros of $G(z + \rho)$ are negative, any sector $\omega_1 \leq \arg z \leq \omega_2 \leq \pi - \omega_1$ containing all the zeros of $A(z)$ will also contain all the zeros of $F(z)$. For example, if $A(z) = (z - 2)(z + 1 - i)$, then $F(z) = 0.5z^2 - (1 + i)z - 2 + 2i$, which has the zeros $(3.058 + 0.514i)$ and $(-1.058 + 1.486i)$, both thus being in the sector $0 \leq \arg z \leq 135^\circ$ containing the zeros of $A(z)$.

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ON THE EXTENSION OF A VECTOR FUNCTION SO AS TO PRESERVE A LIPSCHITZ CONDITION

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1. Introduction. Let $V$ be a two-dimensional Euclidean space, and let $x$ be a vector ranging over $V$. The vector function $f(x)$ is to be a vector in $V$ defined over a set $S$ of the space $V$. The Euclidean distance between any two points $x$ and $y$ in the plane is denoted by $|x - y|$. Furthermore $f(x)$ is to satisfy a Lipschitz condition, so that there exists a positive constant $K$ such that

\[ |f(x_1) - f(x_2)| \leq K |x_1 - x_2| \]

holds for all pairs $x_1$ and $x_2$ in $S$.

In event $f(x)$ is a real-valued function of a variable $x$ ranging over a set $S$ of a metric space, then the extension of the definition of $f(x)$ to any set $T \supset S$ so as to satisfy the condition (1) has been accomplished.\(^1\) The present paper establishes the result that the vector function $f(x)$ can be extended to any set $T \supset S$ so as to satisfy the Lipschitz condition with the same constant $K$. In §3 it is shown how the method used to obtain the above result can be applied to yield an extension for the case considered by McShane.\(^2\) If $f(x)$ has its


\(^2\) Loc. cit.