THE ZEROS OF CERTAIN COMPOSITE POLYNOMIALS

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1. **Introduction.** If $A_0(z)$ is a given mth degree polynomial and

$$(1.1)\quad A_k(z) = (\beta_k - z)A'_{k-1}(z) + (\gamma_k - k)A_{k-1}(z), \qquad \gamma_k \neq m + k, \quad k = 1, 2, \cdots, n,$$

we may obtain various theorems on the relative location of the zeros of $A_0(z)$ and $A_n(z)$ by the familiar method of first finding such relations for two successive $A_k(z)$ and then iterating the relations n times.

This method has already been employed in the study of the zeros of sequence (1.1) for the following three cases: (1) for all k, $\beta_k=0$ and γ_k is real;[1] (2) for all k, $\gamma_k=m+1$—a limiting case leading to Grace's theorem,[2] and (3) the limiting case that for all k, as $h\to 0$, $h\beta_k \to \beta_k'$ and $h(\gamma_k-k)\to 1$, in which case lim $h^kA_k(z)$ is a linear combination of $A_0(z)$ and its first k derivatives.[3]

In the present article we propose to apply the method to the case that *the parameters* β_k *and* γ_k *are complex numbers represented by points within certain given regions of the plane.*

To calculate the nth iterate $A_n(z)$ in our case, let us define

$$(1.2)\qquad A(z) \equiv A_0(z) \equiv a_0 + a_1z + \cdots + a_mz^m;$$

$$(1.3)\qquad \begin{aligned} B(z) &\equiv (\beta_1 - z)(\beta_2 - z)\cdots(\beta_n - z) \\ &\equiv b_0 + b_1z + \cdots + b_nz^n, \end{aligned}$$

$$(1.4)\qquad C(z) \equiv (\gamma_1 - 1 - z)(\gamma_2 - 2 - z)\cdots(\gamma_n - n - z);$$

$$S(z, k, p) \equiv B(z)\sum \frac{\gamma_{j_1}^{(k+p)} - 1}{\beta_{j_1} - z}\cdot\frac{\gamma_{j_2}^{(k+p)} - 2}{\beta_{j_2} - z}\cdots\frac{\gamma_{j_{n-p}}^{(k+p)} - (n-p)}{\beta_{j_{n-p}} - z},$$

where $[\gamma_j^{(r)} \equiv \gamma_j - r]$ thus $\gamma_j^{(r)} - j$ is a zero of $C(z+r)$, $p<n$, and the sum is formed for all j_i such that $1 \leqq j_1 < j_2 < \cdots < j_{n-p} \leqq n$;

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[1] See Laguerre, *Oeuvres*, Paris, 1898, vol. 1 pp. 200–202, and G. Polya, *Ueber einem Satz von Laguerre*, Jber. Deutschen Math. Verein. vol. 38 (1929) pp. 161–168.

[2] See Laguerre, *Oeuvres*, vol 1 p. 49, and G. Szegö, *Bemerkungen zu einem Satz von S. H. Grace*, Math. Zeit. vol. 13 (1922) pp. 28–55, p. 33.

[3] See M. Fujiwara, *Eine Bemerkungen uber die elementare Theorie der algebraischen Gleichungen*, Tôhoku Math. J. vol. 9 (1916) pp. 102–108; T. Takagi, *Note on the algebraic equations*, Proceedings of the Physico-Mathematical Society of Japan vol. 3 (1921) pp. 175–179; J. L. Walsh, *On the location of the roots of polynomials*, Bull. Amer. Math. Soc. vol. 30 (1924) p. 52, and M. Marden, *On the zeros of the derivative of a rational function*, Bull. Amer. Math. Soc. vol. 42 (1936) p. 406.

$$S(z, k, n) \equiv B(z) \quad \text{and} \quad S(z, k, p) \equiv 0 \quad \text{for} \quad p > n.$$

Then by repeated use of formula (1.1), we find for

$$D(z) \equiv A_n(z) \equiv d_0 + d_1 z + \cdots + d_m z^m \tag{1.5}$$

the two expressions

$$\begin{aligned} D(z) &= \sum_{p=0}^{n} S(z, 0, p) \frac{d^p A(z)}{dz^p} \\ D(z) &= \sum_{k=0}^{m} \sum_{p=0}^{m-k} \frac{(k+p)!}{k!} S(0, k, p) a_{k+p} z^k. \end{aligned} \tag{1.6}$$

Let us note two special cases of these formulas. First, if $\beta_k = 0$ for all k, then

$$S(0, k, p) = 0 \quad \text{for} \quad p \neq 0, \qquad S(0, k, 0) = C(k)$$

and, hence,

$$D(z) = C(0)a_0 + C(1)a_1 z + \cdots + C(m)a_m z^m. \tag{1.7}$$

Secondly, if, for all k, $\gamma_k = \gamma + 1$, where γ is any constant other than $m, m+1, \cdots, m+n-1$, then

$$\begin{aligned} S(0, k, p) &= (\gamma - k - p)(\gamma - k - p - 1) \cdots (\gamma + 1 - k - n) \sum \beta_{j_1}\beta_{j_2} \cdots \beta_j \\ &= (-1)^{n-p}(n-p)! C_{\gamma-k-p, n-p} b_{n-p} \end{aligned}$$

where $C_{r,s} = r(r-1) \cdots (r-s+1)/1 \cdot 2 \cdots s$ and, hence, except for the multiplier $n!$,

$$D(z) = \sum_{k=0}^{m} \sum_{p=0}^{m-k} (-1)^{n-p} C_{n,p}^{-1} C_{\gamma-k-p, n-p} C_{k+p, k} a_{k+p} b_{n-p} z^k \tag{1.8}$$

with $b_{n-p} = 0$ for $p > n$.

In what follows it will be convenient to denote by a script capital $\mathcal{F}$ a region containing all the zeros of a given function $F(z)$. Thus, $\mathcal{A}$: $|z| \leqq r$ will mean that all the zeros of the polynomial $A(z)$ lie in or on the circle $|z| = r$.

2. **Zeros of two successive $A_k(z)$.** Using the preceding notation, the following lemma may be stated.

LEMMA. *Let $\gamma_j' = \gamma_j - j$ denote the zeros of $C(z)$. Then,*
(a) *$\mathcal{A}_k$: $r_1 \leqq |z| \leqq r_2$ and $|\beta_k| \leqq \lambda r_1$ imply*

$$
\begin{aligned}
\mathcal{A}_{k+1}\colon r_1 \min\left[1, \frac{|\gamma_k'| - m\lambda}{|\gamma_k' - m|}\right] &\leqq |z| \\
&\leqq r_2 \max\left[1, \frac{|\gamma_k'| + m\lambda}{|\gamma_k' - m|}\right];
\end{aligned}
\tag{2.1}
$$

(b) $\mathcal{A}_k$: $|z| \leqq r$ *and* $|\beta_k| \geqq \lambda r$ *imply*

$$
\mathcal{A}_{k+1}\colon |z| \leqq r \quad \text{and} \quad |z| \geqq r \max\left[1, \frac{m\lambda - |\gamma_k'|}{|m - \gamma_k'|}\right];
\tag{2.2}
$$

(c) $\mathcal{A}_k$: $\omega_1 \leqq \arg z \leqq \omega_2$ *with* $\omega_2 - \omega_1 \leqq \pi$ *and* $\beta_k = 0$ *imply*

$$
\begin{aligned}
\mathcal{A}_{k+1}\colon \omega_1 + \min\left(0, \arg \frac{\gamma_k'}{\gamma_k' - m}\right) &\leqq \arg z \\
&\leqq \omega_2 + \max\left(0, \arg \frac{\gamma_k}{\gamma_k' - m}\right).
\end{aligned}
\tag{2.3}
$$

This lemma may be deduced from the results of a previous paper[4] or may be proved directly as follows.

Let $\mathcal{A}_k$ be a circular region and let ζ be any zero of $A_{k+1}(z)$ outside $\mathcal{A}_k$. Then, by Laguerre's theorem,[5] there exists a point α in $\mathcal{A}_k$ such that $[A_k'(\zeta)/A_k(\zeta)] = m/(\zeta - \alpha)$ and, hence, by (1.1)

$$
\zeta = \frac{\gamma_k' \alpha - m\beta_k}{\gamma_k' - m}.
\tag{2.4}
$$

In particular for $|\beta_k| \leqq \lambda r_1$, if $\mathcal{A}_k$: $|z| \leqq r_2$, then[6] we have that $|\zeta| \leqq r_2(|\gamma_k'| + m\lambda)|\gamma_k' - m|^{-1}$, whereas if $\mathcal{A}_k$: $|z| \geqq r_1$, then $|\zeta| \geqq r_1(|\gamma_k'| - m\lambda)|\gamma_k' - m|^{-1}$. Hence, if all the zeros of $A_k(z)$ lie in the ring $r_1 \leqq |z| \leqq r_2$, an arbitrarily chosen zero of $A_{k+1}(z)$ must lie in the ring (2.1).

If $|\beta_k| \geqq \lambda r$ and $\mathcal{A}_k$: $|z| \leqq r$, then $|\zeta| \geqq r(m\lambda - |\gamma_k'|)|\gamma_k' - m|^{-1}$ and hence the zeros of $A_{k+1}(z)$ not satisfying the first inequality (2.2) must satisfy the second inequality (2.2).

Finally, for $\beta_k = 0$, if $\mathcal{A}_k$: $\omega \leqq \arg z \leqq \omega + \pi$, then $\omega + \arg [\gamma_k'(\gamma_k' - m)^{-1}] \leqq \arg \zeta \leqq \omega + \pi + \arg [\gamma_k'(\gamma_k' - m)^{-1}]$. Setting $\omega = \omega_1$ and $\omega = \omega_2 - \pi$ and combining the results, we conclude that, if all the zeros of $A_k(z)$ lie in the sector $\omega_1 \leqq \arg z \leqq \omega_2$, then all the zeros of $A_{k+1}(z)$ lie in the sector (2.3).

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[4] M. Marden, ibid. pp. 400–401. See also J. L. Walsh, *On the location of the roots of certain types of polynomials*, Trans. Amer. Math. Soc. vol. 24 (1922) p. 169, lemma, and Polya-Szegö, *Aufgaben der Analysis*, Berlin 1925 vol. 2 p. 58, problem 117.

[5] Laguerre, *Oeuvres*, vol. 1 p. 49.

[6] See M. Marden, ibid. p. 402.

3. **Zeros of $A_0(z)$ and $A_n(z)$.** We shall now apply part (1) of the lemma to the successive $A_k(z)$ in order to determine the relative location of the zeros of the polynomials $A(z) \equiv A_0(z)$, $B(z)$, $C(z)$ and $D(z) \equiv A_n(z)$. In addition to the notation used hitherto, we shall use the symbol $\mathfrak{C}(z)$ for the polynomial whose zeros are the moduli of the zeros of $C(z)$:

$$\mathfrak{C}(z) = (|\gamma_1'| - z)(|\gamma_2'| - z) \cdots (|\gamma_n| - z).$$

THEOREM I. *Given the positive constants* ρ *and* λ ($\lambda < 1$). *Then,*

(1) $\mathcal{A}$: $|z| \leqq r$, $\mathcal{B}$: $|z| \leqq \lambda r$ *and* $\mathcal{C}$: $\rho|z-m| \geqq |z| + m\lambda$ *imply* $\mathcal{D}$: $|z| \leqq r \max(1, \rho^n)$;

(2) $\mathcal{A}$: $|z| \leqq r$, $\mathcal{B}$: $|z| \leqq \lambda r$ *and* $\mathcal{C}$: $0 < \rho|z-m| \leqq |z| + m\lambda$ *with* $\rho \geqq 1$ *imply* $\mathcal{D}$: $|z| \leqq r|\mathfrak{C}(-m\lambda)/C(m)|$;

(3) $\mathcal{A}$: $|z| \geqq r$, $\mathcal{B}$: $|z| \leqq \lambda r|\mathfrak{C}(m\lambda)/C(m)|$ *and* $\mathcal{C}$: $\rho|z-m| \geqq |z| - m\lambda > 0$ *with* $\rho \leqq 1$ *imply* $\mathcal{D}$: $|z| \geqq r|\mathfrak{C}(m\lambda)/C(m)|$;

(4) $\mathcal{A}$: $|z| \geqq r$, $\mathcal{B}$: $|z| \leqq \lambda r \min(1, \rho^n)$ *and* $\mathcal{C}$: $0 < \rho|z-m| \leqq |z| - m\lambda$ *imply* $\mathcal{D}$: $|z| \geqq r \min(1, \rho^n)$.

To prove this theorem, let us define

$$\mu_k = |m - \gamma_k'|^{-1}(|\gamma_k'| + m\lambda);$$

$$M_k = \max \mu_1^{\sigma_1}\mu_2^{\sigma_2} \cdots \mu_k^{\sigma_k}, \qquad \text{where } \sigma_j = 0, 1;$$

$$\nu_k = |m - \gamma_k'|^{-1}(|\gamma_k'| - m\lambda) \quad \text{if} \quad |\gamma_k'| > m\lambda \quad \text{and}$$

$$\nu_k = 0 \qquad \text{if} \quad |\gamma_k'| \leqq \lambda m;$$

$$N_k = \min \nu_1^{\sigma_1}\nu_2^{\sigma_2} \cdots \nu_k^{\sigma_k}, \qquad \text{where } \sigma_j = 0, 1.$$

If $\mathcal{A}$: $|z| \leqq r$ and $\mathcal{B}$: $|z| \leqq \lambda r$, then by the right side of (2.1)

$$\mathcal{A}_1: |z| \leqq rM_1, \quad \mathcal{A}_2: |z| \leqq rM_2, \cdots, \mathcal{A}_n: |z| \leqq rM_n.$$

Since in part (1) of Theorem I

$$\mu_k \leqq \rho,$$

$M_n = \max(1, \rho^n)$, and, since in part (2) $\mu_k \geqq 1$,

$$M_n = \mu_1\mu_2 \cdots \mu_n = |\mathfrak{C}(-m\lambda)/C(m)|.$$

If $\mathcal{A}$: $|z| \geqq r$ and $\mathcal{B}$: $|z| \leqq \lambda r N_n$, then by the left side of (2.1)

$$\mathcal{A}_1: |z| \geqq rN_1, \quad \mathcal{A}_2: |z| \geqq rN_2, \cdots, \mathcal{A}_n: |z| \geqq rN_n.$$

Since in part (3) of Theorem I $0 < \nu_k \leqq \rho \leqq 1$, $N_n = \nu_1\nu_2 \cdots \nu_n = |\mathfrak{C}(m\lambda)/C(m)|$; whereas since in part (4) $\nu_k \geqq \rho$, $N_n = \min(1, \rho^n)$. We have thus established Theorem I.

It is to be noticed that each region $\mathcal{C}$ of Theorem I is bounded by one of the ovals $\rho|m-z|=|z|\pm m\lambda$ of the cartesian curve[7]

$$(3.1)\quad [(\rho^2-1)(x^2+y^2)-2m\rho^2x+m^2(\rho^2-\lambda^2)]^2=4m^2\lambda^2(x^2+y^2)$$

having ordinary foci at the three points $z=0$, $z=m$ and $z=m(\rho^2-1)^{-1}(\rho^2-\lambda^2)$ and a singular focus at the point $z=m\rho^2(\rho^2-1)^{-1}$. If $\rho>1$, curve (3.1) consists of two nested ovals both enclosing $z=m$ and both excluding $z=0$; in this case, the region $\mathcal{C}$ of part (1) of the theorem is the exterior of the outer oval, $\mathcal{C}$ of part (2) is the interior of the outer oval exclusive of point $z=m$ and $\mathcal{C}$ of part (4) is the interior of the inner oval exclusive of point $z=m$. If $\rho=1$, curve (3.1) degenerates into the hyperbola with foci at $z=0$ and $z=m$ and transverse axis of $m\lambda$; in this case $\mathcal{C}$ of part (1) is the region left of the left branch of the hyperbola, $\mathcal{C}$ of part (2) is the region right of the left branch not including $z=m$, $\mathcal{C}$ of part (3) is the region common to the exterior of circle $|z|=m\lambda$ and the left of the right branch and $\mathcal{C}$ of part (4) is the interior of the right branch with point $z=m$ omitted. If $\lambda<\rho<1$, curve (3.1) consists of nested ovals, now however both containing $z=0$ and excluding $z=m$; in this case, $\mathcal{C}$ of part (1) is the interior of the inner oval, $\mathcal{C}$ of part (3) is the region common to the exterior of circle $|z|=m\lambda$ and the interior of the outer oval and $\mathcal{C}$ of part (4) is the exterior of the outer oval exclusive of point $z=m$. In the latter case, if $\rho\to\lambda$, the inner oval shrinks to a point and hence, for $\rho<\lambda$, $\mathcal{C}$ of part (1) is a null-set, and the $\mathcal{C}$'s of parts (3) and (4) are those described for $\lambda<\rho<1$.

In the foregoing discussion, we have implied that $\lambda\neq 0$. If $\lambda=0$, curve (3.1) degenerates into the dipolar circle $\rho|z-m|=|z|$ and $D(z)$ is given by formula (1.7). We may thus state the following corollary.

COROLLARY. *If all the zeros of a polynomial $A(z)=a_0+a_1z+\cdots+a_mz^m$ lie in the ring $0\leqq r_1\leqq|z|\leqq r_2\leqq\infty$ and if all the zeros of an nth degree polynomial $C(z)$ lie in the connected region bounded by the circles $|z|=\rho_1|z-m|$ and $|z|=\rho_2|z-m|$ with $\rho_1\leqq\rho_2$, then all the zeros of the polynomial $D(z)=C(0)a_0+C(1)a_1z+\cdots+C(m)a_mz^m$ lie in the ring*[8]

$$(3.2)\qquad r_1\min(1,\rho_1^n)\leqq|z|\leqq r_2\max(1,\rho_2^n).$$

If $\rho_2<1$, the left side of (3.2) *may be replaced by the then larger number*

[7] See G. Loria, *Curve piane speciali*, Milan, 1930, vol. I pp. 212–214.

[8] For the cases (1) $r_1=0$, $\rho_1=0$, $\rho_2=1$; (2) $r_2=\infty$, $\rho_1=1$, $\rho_2=\infty$; and (3) $r_1=r_2$, $\rho_1=\rho_2=1$, see N. Obrechkoff, *Sur les zeros des polynômes*, C. R. Acad. Sci. Paris vol. 209 (1939) pp. 1270–1272, and L. Weisner, *Roots of certain classes of polynomials*, Bull. Amer. Math. Soc. vol. 48 (1942) p. 283–286.

$r_1|C(0)/C(m)|$ *and, if* $1<\rho_1$, *the right side may be replaced by the then smaller number* $r_2|C(0)/C(m)|$.

So far we have applied part (1) of the lemma to the successive $A_k(z)$. Similarly, if we apply part (3) of the lemma and formula (1.7), we may obtain the following result.

THEOREM II. *If all the zeros of the polynomial* $A(z)=a_0+a_1z+\cdots+a_mz^m$ *are in the sector* $\omega_1\leqq \arg z\leqq\omega_2$ *with* $\omega_2-\omega_1=\omega\leqq\pi$, *and if all the zeros of an nth degree polynomial* $C(z)$ *are in the lune* $\theta_1\leqq\arg[z/(z-m)]\leqq\theta_2$ *with* $|\theta_1|+|\theta_2|\leqq(\pi-\omega)/n$, *then all the zeros of the polynomial* $D(z)=a_0C(0)+a_1C(1)z+\cdots+a_mC(m)z^m$ *lie in the sector*

$$\omega_1+\min(0,n\theta_1)\leqq\arg z\leqq\omega_2+\max(0,n\theta_2). \tag{3.3}$$

If $\theta_2<0$, $\min(0,n\theta_1)$ *may be replaced in* (3.3) *by the then larger number* $\arg C(0)/C(m)$ *and, if* $0<\theta_1$, $\max(0,n\theta_2)$ *may be replaced by the then smaller number* $\arg C(0)/C(m)$.

4. **Entire functions.** Theorem II and the corollary to Theorem I may be generalized at once through replacing

$$\begin{aligned} D(z) &= a_0C(0)+a_1C(1)z+\cdots+a_mC(m)z^m \\ &= \delta(\delta_1-z)(\delta_2-z)\cdots(\delta_m-z) \end{aligned}$$

by

$$F(z)=a_0E(0)+a_1E(1)z+\cdots+a_mE(m)z^m,$$

where

$$E(z)=e^{\lambda z}C(z)\quad\text{and}\quad\lambda=\mu+i\nu.$$

In fact, since

$$F(z)=\sum_{k=0}^m a_kC(k)e^{\lambda k}z^k=D(e^\lambda z)=\delta e^{m\lambda}\prod_{k=1}^m(\delta_ke^{-\lambda}-z),$$

the substitution of $E(z)$ and $F(z)$ for $C(z)$ and $D(z)$ would require only the following changes: in the corollary to Theorem I, inequality (3.2) becomes

$$e^{-\mu}r_1\min(1,\rho_1^n)\leqq|z|\leqq e^{-\mu}r_2\max(1,\rho_2^n) \tag{4.1}$$

where $e^{m\mu}|E(0)/E(m)|$ may replace $\min(1,\rho_1^n)$ if $\rho_2\leqq1$ and $\max(1,\rho_2^n)$ if $\rho_1\leqq1$; in Theorem II, inequality (3.3) becomes

$$\omega_1-\nu+\min(0,n\theta_1)\leqq\arg z\leqq\omega_2-\nu+\max(0,n\theta_2) \tag{4.2}$$

where $[m\nu+\arg E(0)/E(m)]$ may replace min $(0, n\theta_1)$ if $\theta_2 \leqq 0$ and max $(0, n\theta_2)$ if $0<\theta_1$.

Furthermore, these results may be extended to entire functions $E(z)$ of genus zero or one provided the zeros of $E(z)$ are assumed to lie in infinite regions, determined by taking $\rho_1=1$ or $\rho_2=1$ and $\theta_1=\theta_2=0$.

THEOREM III. *Given the entire functions*

$$A(z) = \sum_{k=0}^{m} a_k z^k, \qquad E(z) = e^{\lambda_0 z}\prod_{k=1}^{\infty}\left(1 - \frac{z}{\gamma_k}\right)e^{\lambda_k z},$$

$$F(z) = \sum_{k=0}^{m} a_k E(k) z^k,$$

where $\lambda_j=\mu_j+i\nu_j$.

(a) *If all the zeros of* $A(z)$ *lie in the ring* $0\leqq r_1\leqq |z| \leqq r_2 \leqq \infty$, *if all the zeros of* $E(z)$ *lie in the region* $\rho_1 \leqq |z/(z-m)| \leqq \rho_2$ *with at least one number* ρ_1, ρ_2 *unity, and if* $\mu_0+\mu_1+\cdots\rightarrow\mu$, *then all the zeros of* $F(z)$ *lie in the ring* $K_1e^{-\mu}r_1\leqq |z| \leqq K_2e^{-\mu}r_2$, *where* $K_1=1$ *or* $e^{\mu m}|E(0)/E(m)|$ *according as* $\rho_1=1$ *or* $\rho_1<1$ *and* $K_2=1$ *or* $e^{\mu m}|E(0)/E(m)|$ *according as* $1=\rho_2$ *or* $1<\rho_2$.

(b) *If all the zeros of* $A(z)$ *lie in the sector* $\omega_1\leqq\arg z\leqq\omega_2$ *with* $\omega_2-\omega_1\leqq\pi$, *if all the zeros of* $E(z)$ *lie on the real axis outside of the segment* $(0, m)$ *and if* $\nu_0+\nu_1+\cdots\rightarrow\nu$, *then all the zeros of* $F(z)$ *lie in the sector* $\omega_1-\nu\leqq\arg z\leqq\omega_2-\nu$.

Theorem III(b) is a partial generalization of results due to Laguerre and Polya[1] in the case that both $\nu=0$ and all the zeros of $A(z)$ are real. However, it may also be derived from this special case by use of the theorem quoted in problem 153, p. 65, vol. 2 Polya-Szegö's *Aufgaben der Analysis*. For this fact and its following proof, the author is indebted to the referee, Professor Polya.

We may assume without loss of generality that $\nu=0$. Then, according to the Laguerre-Polya results, $\alpha_k=E(k)$ form a set of multipliers such that, if any polynomial $A(z)=a_0+a_1z+\cdots+a_mz^m$ has only positive (negative) zeros, so has also the polynomial $C(z)=\alpha_0a_0+\alpha_1a_1z+\cdots+\alpha_ma_mz^m$. But such multipliers have also the property that, if all the zeros of $A(z)$ lie in the sector $\omega_1\leqq\arg z\leqq\omega_2$ with $\omega_2-\omega_1\leqq\pi$, all the zeros of $C(z)$ also lie in this sector. For, since all the zeros of $(1+z)^m$ are negative, the zeros of polynomial

$$G(z) = \alpha_0 + C_{m,1}\alpha_1 z + C_{m,2}\alpha_2 z^2 + \cdots + \alpha_m z^m$$

are also all negative, and, since the sector is a convex region containing the origin, the theorem from Polya-Szegö may be applied with the $F(z)$ of the theorem taken as $A(z)$. Theorem III(b) then follows immediately.

As an application of Theorem III, let us consider the polynomial $F(z)=\sum_{k=0}^{m}a_kG(k+p)z^k$ where $p>0$ and $G(z)=\Gamma(z)^{-1}=e^{\mu z}\prod_{n=1}^{\infty}(1+n^{-1}z)e^{-z/n}$, the reciprocal of the gamma function. Since $\nu=0$ and all the zeros of $G(z+p)$ are negative, any sector $\omega_1\leqq\arg z\leqq\omega_2\leqq\pi-\omega_1$ containing all the zeros of $A(z)$ will also contain all the zeros of $F(z)$. For example, if $A(z)=(z-2)(z+1-i)$, then $F(z)=0.5z^2-(1+i)z-2+2i$, which has the zeros $(3.058+0.514i)$ and $(-1.058+1.486i)$, both thus being in the sector $0\leqq\arg z\leqq 135°$ containing the zeros of $A(z)$.

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ON THE EXTENSION OF A VECTOR FUNCTION SO AS TO PRESERVE A LIPSCHITZ CONDITION

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1. Introduction. Let V be a two-dimensional Euclidean space, and let x be a vector ranging over V. The vector function $f(x)$ is to be a vector in V defined over a set S of the space V. The Euclidean distance between any two points x and y in the plane is denoted by $|x-y|$. Furthermore $f(x)$ is to satisfy a Lipschitz condition, so that there exists a positive constant K such that

$$|f(x_1)-f(x_2)|\leqq K|x_1-x_2| \tag{1}$$

holds for all pairs x_1 and x_2 in S.

In event $f(x)$ is a real-valued function of a variable x ranging over a set S of a metric space, then the extension of the definition of $f(x)$ to any set $T\supset S$ so as to satisfy the condition (1) has been accomplished.[1] The present paper establishes the result that the *vector* function $f(x)$ can be extended to any set $T\supset S$ so as to satisfy the Lipschitz condition with the same constant K. In §3 it is shown how the method used to obtain the above result can be applied to yield an extension for the case considered by McShane.[2] If $f(x)$ has its

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[1] E. J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 837–842.

[2] Loc. cit.