

ing the origin, the theorem from Polya-Szegö may be applied with the $F(z)$ of the theorem taken as $A(z)$. Theorem III(b) then follows immediately.

As an application of Theorem III, let us consider the polynomial $F(z) = \sum_{k=0}^m a_k G(k+p)z^k$ where $p > 0$ and $G(z) = \Gamma(z)^{-1} = e^{\mu z} \prod_{n=1}^{\infty} (1+n^{-1}z)e^{-z/n}$, the reciprocal of the gamma function. Since $\nu=0$ and all the zeros of $G(z+p)$ are negative, any sector $\omega_1 \leq \arg z \leq \omega_2 \leq \pi - \omega_1$ containing all the zeros of $A(z)$ will also contain all the zeros of $F(z)$. For example, if $A(z) = (z-2)(z+1-i)$, then $F(z) = 0.5z^2 - (1+i)z - 2 + 2i$, which has the zeros $(3.058 + 0.514i)$ and $(-1.058 + 1.486i)$, both thus being in the sector $0 \leq \arg z \leq 135^\circ$ containing the zeros of $A(z)$.

UNIVERSITY OF WISCONSIN AT MILWAUKEE

ON THE EXTENSION OF A VECTOR FUNCTION SO AS TO PRESERVE A LIPSCHITZ CONDITION

F. A. VALENTINE

1. Introduction. Let V be a two-dimensional Euclidean space, and let x be a vector ranging over V . The vector function $f(x)$ is to be a vector in V defined over a set S of the space V . The Euclidean distance between any two points x and y in the plane is denoted by $|x-y|$. Furthermore $f(x)$ is to satisfy a Lipschitz condition, so that there exists a positive constant K such that

$$(1) \quad |f(x_1) - f(x_2)| \leq K |x_1 - x_2|$$

holds for all pairs x_1 and x_2 in S .

In event $f(x)$ is a real-valued function of a variable x ranging over a set S of a metric space, then the extension of the definition of $f(x)$ to any set $T \supset S$ so as to satisfy the condition (1) has been accomplished.¹ The present paper establishes the result that the *vector* function $f(x)$ can be extended to any set $T \supset S$ so as to satisfy the Lipschitz condition with the same constant K . In §3 it is shown how the method used to obtain the above result can be applied to yield an extension for the case considered by McShane.² If $f(x)$ has its

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¹ E. J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 837-842.

² Loc. cit.

functional values in a metric space then, in general, it cannot be extended so as to preserve (1). The geometric theorems which arise in forming the extension prove to be of as much interest as the extension itself. In the following section it is shown that this extension is a consequence of the following theorem about sets of circles in the plane. A circle with center x_i and radius r_i is the two-dimensional set of points x for which $|x - x_i| \leq r_i$.

THEOREM 1. *Consider in the plane two sets of circles M and M' such that to each circle in M there corresponds a circle in M' . Let $C'_i \in M'$, having center x'_i and radius r'_i , correspond to $C_i \in M$, having center y_i and radius r_i .*

Suppose that

$$r'_i = r_i$$

for all corresponding circles C_i and C'_i . Furthermore suppose that

$$(2) \quad |x'_i - x'_j| \leq |y_i - y_j|$$

for all corresponding pairs of circles (C_i, C_j) and (C'_i, C'_j) . Finally suppose that the intersection of all the circles in M is not empty, so that the product

$$(3) \quad \prod_M C_i \neq 0.$$

Then it is true that the intersection of all the circles in M' is such that

$$(4) \quad \prod_{M'} C'_i \neq 0.$$

2. The extension. If the set S consists of a single point x_1 , the extension of $f(x)$ to a second point x_2 is trivial. We will let S' be the set in the f -plane which corresponds to the set S in the x -plane.

Suppose that the set S consists of two points x_1 and x_2 , and that we wish to extend $f(x)$ to a third point x_3 . With the points Kx_j as centers ($j = 1, 2$) draw circles C_j with radii $r_j = K|x_j - x_3|$, respectively. Define x'_j to be

$$(5) \quad x'_j \equiv f(x_j), \quad j = 1, 2.$$

With x'_j as centers draw circles C'_j with radii $r'_j = r_j$, respectively. Since by hypothesis $|x'_1 - x'_2| \leq K|x_1 - x_2|$, and since by construction the product $C_1 \cdot C_2 \neq 0$, it is true that $C'_1 \cdot C'_2 \neq 0$. To extend $f(x)$ to x_3 so as to satisfy (1), choose $x'_3 \in C'_1 \cdot C'_2$. Since $|x'_j - x'_3| \leq r'_j = r_j = K|x_j - x_3|$, letting $f(x_3) \equiv x'_3$, the function $f(x)$ has been extended from $S = (x_1, x_2)$ to $T = (x_1, x_2, x_3)$ so as to satisfy the Lipschitz condition (1).

The crucial extension, as we shall see, is from $S = (x_1, x_2, x_3)$ to $T = (x_1, x_2, x_3, x_4)$. In order to construct this extension, the following lemma and theorem are essential. For convenience denote the triangular set determined by three points y_1, y_2, y_3 by $\Delta(y_1, y_2, y_3)$.

LEMMA 1. *If three circles C_i' ($i = 1, 2, 3$) with centers x_i' , respectively, are such that*

$$(6) \quad C_i' \cdot C_j' \cdot \Delta(x_i', x_j', x_k') \neq 0, \quad i, j = 1, 2, 3,$$

$$(7) \quad C_1' \cdot C_2' \cdot C_3' \cdot \Delta(x_1', x_2', x_3') = 0,$$

then it is true that³

$$(8) \quad B(C_i') \cdot B(C_j') \cdot \Delta(x_i', x_j', x_k') \neq 0, \quad i, j = 1, 2, 3,$$

where $B(C_i')$ is the boundary of the circle C_i' .

To prove Lemma 1 select three points $x_{ij} \in C_i' \cdot C_j' \cdot \Delta$ ($i = 1, 2; j = 2, 3; j > i$) which exist by virtue of (6). Since $\Delta(x'_{12}, x'_{13}, x'_{23}) \subset \Delta(x_1', x_2', x_3')$, in proving (8) it is sufficient to prove

$$(9) \quad B(C_i') \cdot B(C_j') \cdot \Delta(x'_{12}, x'_{13}, x'_{23}) \neq 0.$$

Denote the line segment joining points x and y by \overline{xy} . Since C_1' is convex, $\overline{x'_{12}x'_{13}} \subset C_1'$. Also (7) implies that the point x'_{23} is not contained in C_1' . Hence the circle C_1' contains a maximum subsegment $\overline{x'_{12}P_2}$ of $\overline{x'_{12}x'_{23}}$ and a maximum subsegment $\overline{x'_{13}P_3}$ of $\overline{x'_{13}x'_{23}}$, where the points $P_2 \neq P_3 \neq x'_{23}$. Since $\overline{P_2P_3} \subset C_1'$, since $x'_{23} \notin C_1'$, and since $B(C_1')$ cannot intersect the $B[\Delta(P_2, P_3, x'_{23})]$ except at P_2 and P_3 , a simple arc of $B(C_1')$ joins P_2 and P_3 and is contained in $\Delta(P_2, P_3, x'_{23})$ and hence in $\Delta(x'_{12}, x'_{13}, x'_{23})$.

In exactly the same way, there exist points $Q_3 \in \overline{x'_{13}x'_{23}}$, $Q_1 \in \overline{x'_{12}x'_{13}}$, with $Q_1 \neq Q_3 \neq x'_{13}$, such that Q_1 and Q_3 are joined by a simple arc of $B(C_2')$ contained in $\Delta(x'_{12}, x'_{13}, x'_{23})$. Suppose the point P_3 lay between x'_{23} and Q_3 on $\overline{x'_{23}x'_{13}}$. This supposition, together with the facts $C_2' \supset \overline{x'_{23}Q_3}$, $C_1' \supset \overline{x'_{13}P_3}$, implies that $\overline{x'_{23}x'_{13}} \subset C_1' + C_2'$. But since $\overline{x'_{23}x'_{13}} \subset C_3'$, the previous statement implies that $C_1' \cdot C_2' \cdot C_3' \cdot \Delta \neq 0$, contrary to assumption (7). Hence P_3 is not between x'_{23} and Q_3 on

³ Lemma 1 is stated for circles in order to shorten the proof of Theorem 2; however the lemma can be generalized as follows: Suppose that each of the sets C_1, C_2, C_3, Δ and each of the products $C_i \cdot \Delta, C_i \cdot C_j \cdot \Delta$ ($i, j = 1, 2, 3$) is a simply connected plane set having a simple closed curve as its boundary. Furthermore suppose $C_1 \cdot C_2 \cdot C_3 = 0, C_i \cdot C_j \cdot B(\Delta) \neq 0$, where $B(\Delta)$ is the boundary of Δ . Then it is true that $B(C_i) \cdot B(C_j) \cdot \Delta \neq 0$.

$\overline{x'_{23}x'_{13}}$, so that $Q_3 \in \overline{x'_{23}x'_{13}} - \overline{x'_{13}P_3}$. This implies that the points P_2, Q_3, P_3, Q_1 lie on the boundary of $\Delta(x'_{12}, x'_{13}, x'_{23})$ in a clockwise order in the order just given.

Since P_2 and P_3 are joined by a simple arc of $B(C'_1)$ contained in $\Delta(x'_{12}, x'_{13}, x'_{23})$, and since Q_1 and Q_3 are joined by an arc of $B(C'_2)$ having the same properties, condition (9) holds with $i=1, j=2$. Since we can choose any pair of the circles C'_i and repeat the above argument, condition (9) has been proved, and hence Lemma 1 is established.

THEOREM 2. *Suppose the two triangles $\Delta(y_1, y_2, y_3)$ and $\Delta(x'_1, x'_2, x'_3)$ are such that*

$$(10) \quad |x'_i - x'_j| \leq |y_i - y_j|, \quad i, j = 1, 2, 3.$$

Then to any point y_4 in the plane there corresponds a point x'_4 contained in $\Delta(x'_1, x'_2, x'_3)$ such that

$$(11) \quad |x'_4 - x'_i| \leq |y_4 - y_i|, \quad i = 1, 2, 3.$$

To prove Theorem 2, with y_i as centers draw circles C_i with radii $r_i = |y_4 - y_i|$, respectively. Similarly with x'_i as centers draw circles C'_i with radii $r'_i = r_i$, respectively. Since $y_4 \in C_i$, the products $C_i \cdot C_j \cdot \Delta(y_1, y_2, y_3) \neq 0$. Hence condition (10), together with the fact $r'_i = r_i$, implies condition (6).

We first prove that condition (10) implies $C'_1 \cdot C'_2 \cdot C'_3 \cdot \Delta(x'_1, x'_2, x'_3) \neq 0$. Suppose this were not so. Let P_{ij} be a point of intersection of $B(C'_i)$ and $B(C'_j)$. Since we suppose (7) holds, Lemma 1 implies that P_{ij} can be chosen so that $P_{ij} \in \Delta(x'_1, x'_2, x'_3)$ ($i=1, 2; j=2, 3; j>i$). Since we supposed $C'_1 \cdot C'_2 \cdot C'_3 \cdot \Delta = 0$, we have

$$(12) \quad |x'_2 - P_{12}| < |x'_2 - P_{13}|.$$

Denote the angle less than or equal to π , determined by three points a_1, a_2, a_3 , with vertex at a_2 , by $\angle a_1 a_2 a_3$. Since by definition $|x'_1 - P_{12}| = |x'_1 - P_{13}| = r_1$, condition (12) implies by the law of cosines that

$$(13) \quad \angle x'_2 x'_1 P_{12} < \angle x'_2 x'_1 P_{13}.$$

Condition (13), together with the fact $P_{13} \in \Delta(x'_1, x'_2, x'_3)$, implies the first of the inequalities

$$(14) \quad \begin{aligned} \angle x'_2 x'_1 P_{12} + \angle x'_3 x'_1 P_{13} &< \angle x'_2 x'_1 x'_3, \\ \angle x'_1 x'_3 P_{13} + \angle x'_2 x'_3 P_{23} &< \angle x'_1 x'_3 x'_2, \\ \angle x'_3 x'_2 P_{23} + \angle x'_1 x'_2 P_{12} &< \angle x'_3 x'_2 x'_1. \end{aligned}$$

The last two inequalities are proved exactly as the first.

The sum of all the angles in the three triangles $\Delta(x'_i, P_{ij}, x'_j)$ ($i=1, 2; j=2, 3; j>i$) is 3π . By adding the inequalities in (14), we see that in these three triangles $\Delta(x'_i, P_{ij}, x'_j)$ the sum of those six angles, each of which has its vertex at one of the points x'_1, x'_2, x'_3 is less than π . Hence the sum of the remaining three angles in $\Delta(x'_i, P_{ij}, x'_j)$ satisfies the condition

$$(15) \quad \sum_{i,j} \angle x'_i P_{ij} x'_j > 2\pi, \quad i = 1, 2; j = 2, 3; j > i.$$

Furthermore for the $\Delta(y_1, y_2, y_3)$ of Theorem 2 we have

$$(16) \quad \sum_{i,j} \angle y_i y_4 y_j \leq 2\pi, \quad i = 1, 2; j = 2, 3; j > i,$$

the equality holding if $y_4 \in \Delta(y_1, y_2, y_3)$; otherwise the inequality holds. Consequently at least one of the three angles in (15) is greater than the corresponding angle in (16). Without loss of generality renumber the angles so that

$$(17) \quad \angle x'_1 P_{12} x'_2 > \angle y_1 y_4 y_2.$$

By construction $|x'_i - P_{12}| = |y_i - y_4|$ ($i=1, 2$). Condition (17) implies by the law of cosines that in the triangles $\Delta(x'_1, P_{12}, x'_2)$ and $\Delta(y'_1, y'_4, y_2)$

$$(18) \quad |x'_1 - x'_2| > |y_2 - y_1|.$$

This contradicts assumption (10). Hence the assumption that (7) holds is false. Thus choosing a point $x'_4 \in C'_1 \cdot C'_2 \cdot C'_3 \cdot \Delta \neq 0$, we have $|x'_i - x'_i| \leq r'_i = r_i = |y_i - y_4|$ ($i=1, 2, 3$) and Theorem 2 has been established.

We now readily establish the following theorem.

THEOREM 3. *If the plane vector $f(x)$ is defined on a plane set $S = (x_1, x_2, x_3)$ so as to satisfy the Lipschitz condition (1), then it can be extended to any plane set $T = (x_1, x_2, x_3, x_4)$ so as to be Lipschitz preserving. The extension $f(x_4)$ can be defined so as to be contained in the triangle formed by the points $f(x_i)$ ($i=1, 2, 3$).*

PROOF. In Theorem 2, let

$$y_i \equiv Kx_i, \quad x'_i \equiv f(x_i), \quad y_4 \equiv Kx_4, \quad i = 1, 2, 3.$$

Then the point x'_4 in Theorem 2 is the desired extension $f(x_4) \equiv x'_4$, since conditions (10) and (11) imply that

$$|f(x_j) - f(x_k)| \leq K |x_j - x_k|, \quad j, k = 1, \dots, 4.$$

In order to extend $f(x)$ from an arbitrary set S to any set $T \supset S$ we first prove Theorem 1. This can be accomplished by means of Theorem 2 and by a theorem of Helly⁴ as follows. Choose an arbitrary but fixed set of three circles C_i ($i = 1, 2, 3$) in the set of circles M of Theorem 1. Since in Theorem 1 the hypotheses include the fact that $\prod_M C_i \neq 0$, choose the arbitrary point y_4 in Theorem 2 so that $y_4 \in \prod_{i=1}^3 C_i$. Then Theorem 2 implies that there exists a point x'_4 such that $|x'_4 - x'_i| \leq |y_4 - y_i|$. Since $y_4 \in \prod_{i=1}^3 C_i$, it is true that $|y_4 - y_i| \leq r_i$, whence $|x'_4 - x'_i| \leq r_i$. Since $r'_i = r_i$ in Theorem 1, the above inequalities imply that each set of three circles in M' has a point common to all three circles. Now by Helly's theorem, with $n = 2$, it is true that all the circles in M' have a point in common. Thus Theorem 1 is established.

Now we are in a position to prove the following theorem.

THEOREM 4. *Suppose the plane vector function $f(x)$ is defined and satisfies the Lipschitz condition (1) on a set S of the plane.*

Then if T is any set containing S , it is true that $f(x)$ can be extended to T so as to preserve the Lipschitz condition (1). The extension of $f(x)$ can be defined so that the set of points $(f(x), \text{ for } x \in T)$ is contained in any prescribed closed convex set containing S' , where $S' \equiv (f(x), \text{ for } x \in S)$.

PROOF. First, we prove that if U is any set on which $f(x)$ satisfies the Lipschitz condition (1), and if x_0 is an arbitrary point exterior to U , then $f(x)$ can be extended from U to $U + x_0$ so as to satisfy (1). To do this let x_i be an arbitrary point in U , and let

$$x'_i \equiv f(x_i)$$

be the corresponding point in $U' \equiv [f(x), x \in U]$. With $y_i = Kx_i$ as center and with radius $r_i \equiv K|x_0 - x_i|$ draw a circle C_i . As x_i ranges over U , denote the set of circles thus defined by M . Similarly with x'_i as center and with radius $r'_i = r_i$ draw a circle C'_i , and denote the set of all circles, as x'_i ranges over U' , by M' . Since condition (1) holds on U , condition (2) in Theorem 1 holds on U . Furthermore since by construction $r_i = K|x_0 - x_i|$ we have $\prod_M C_i \neq 0$. Hence Theorem 1 implies that $\prod_{M'} C'_i \neq 0$. Consequently there exists a point x'_0 , with $x'_0 \in \prod_{M'} C'_i$, which implies that

$$|x'_0 - x'_i| \leq r_i \equiv K|x_0 - x_i|$$

⁴ E. Helly, Jber. Deutschen Math. Verein vol. 32 (1923) pp. 175-176. The theorem states: *If each $n+1$ sets of a family of closed, bounded, convex sets of the n -dimensional Euclidean space intersect, then there is a point common to all the sets.*

for all corresponding pairs x_i and x'_i in U and U' , respectively. Hence letting $f(x_0) \equiv x'_0$, we have extended $f(x)$ from U to $U+x_0$ so that the Lipschitz condition (1) is preserved. *For convenience denote the above extension from an arbitrary set U to the set $U+x_0$, where x_0 is arbitrary, as extension E.*

The extension from a set S to a region $T \supset S$ of Theorem 4 can be accomplished as follows. Let R be the set of all the points in the plane which have rational coordinates. Extension E of the preceding paragraph implies by *ordinary induction* that $f(x)$ can be extended from the set S to the set $S+R$ so as to satisfy the Lipschitz condition (1). Let \bar{x} be any other point in the plane. In order to effect the extension to \bar{x} let x_i be any sequence of points with rational coordinates such that $\lim_{i \rightarrow \infty} x_i = \bar{x}$. If we define $U \equiv S + \sum x_i$, $x_0 \equiv \bar{x}$, then extension E of the preceding paragraph yields an extension $f(\bar{x})$ so that $f(x)$ satisfies (1) on $U+x_0$. Condition (1) implies that $\lim_{i \rightarrow \infty} f(x_i) = f(\bar{x})$. The function $f(x)$ is defined *uniquely* at \bar{x} , for let y_i be any other sequence of rational points such that $\lim_{i \rightarrow \infty} y_i = \bar{x}$. Letting $U \equiv S + \sum y_i$, and $x_0 \equiv \bar{x}$, extension E yields an extension $f^*(\bar{x})$ such that $f(x)$ satisfies (1) on $U+x_0$. Since

$$|f(y_i) - f(x_i)| \leq K |y_i - x_i|,$$

and since $\lim_{i \rightarrow \infty} f(x_i) = f(\bar{x})$, it follows that

$$(19) \quad \lim_{i \rightarrow \infty} f(y_i) = \lim_{i \rightarrow \infty} f(x_i) = f^*(\bar{x}) = f(\bar{x}).$$

The function $f(x)$ thus defined satisfies the Lipschitz condition (1) on the whole plane. To prove this statement, let x and y be any two points in the plane, and let x_i and y_i be two sequences of rational points such that $\lim_{i \rightarrow \infty} x_i = x$, $\lim_{i \rightarrow \infty} y_i = y$. Then the inequalities

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_i)| + |f(x_i) - f(y_i)| + |f(y_i) - f(y)| \\ &\leq K |x - x_i| + K |x_i - y_i| + K |y_i - y| \end{aligned}$$

imply, by passage to the limit, that

$$|f(x) - f(y)| \leq K |x - y|.$$

Thus the first part of Theorem 4 is proved since if we can extend $f(x)$ to the whole plane we can surely extend it to $T \supset S$.

To prove the last sentence of Theorem 4, let L be any closed convex set such that $L \supset S'$. Consider the sets of circles M and M' defined in the first paragraph following Theorem 4, however with $U = S$ and $U' = S'$. Consider *any* two circles C'_1 and C'_2 in M' . Since the straight line joining the centers of these circles is in L , and since (1) implies

that $C'_1 \cdot C'_2 \neq 0$, we have $C'_1 \cdot C'_2 \cdot L \neq 0$. Furthermore Theorem 1 implies that $\prod_{M'} C'_i \neq 0$. Hence the theorem of Helly⁵ implies that the product

$$L \cdot \prod_{M'} C'_i \neq 0.$$

Hence in extension E one can choose $x'_0 \in L$, whence $S' + x'_0 \subset L$. Now by ordinary induction the function $f(x)$ can be extended to the set $S + R$ (where R is the set of points with rational coordinates), so that the set $S' + R' \equiv [f(x), x \text{ ranges over } S + R] \subset L$. Now the extension of $f(x)$ to the whole plane so that $f(x) \subset L$, x arbitrary, can be constructed. For consider the point \bar{x} in equation (19), where $x_i \subset R$. Since by (19) $\lim_{i \rightarrow \infty} f(x_i) = f(\bar{x})$, since $f(x_i) \in L$, and since L is closed, it follows that $f(\bar{x}) \in L$. Thus the proof of Theorem 4 has been completed.

3. Generalizations. The functions studied by McShane⁶ can be easily extended by the method of the preceding section. Let the *one-dimensional real-valued* function $f(x)$ be defined over a set S of a metric space. Furthermore suppose there exists a real-valued function $w(t) \geq 0$ defined for $0 \leq t$, and satisfying the conditions

$$(20) \quad w(a) + w(b) \geq w(a + b),$$

$$(21) \quad w(b) \geq w(a), \quad \text{for } b \geq a.$$

If the function $f(x)$ satisfies the condition

$$(22) \quad |f(x_1) - f(x_2)| \leq w(\|x_1, x_2\|)$$

for all pairs (x_1, x_2) in S , then $f(x)$ can be extended to the entire space so as to preserve condition (22).

It should be noted that condition (20) does *not* imply that $w(t)$ be concave downward; however, any function $w(t) \geq 0$, $t \geq 0$, which is concave downward must satisfy condition (20). The function $w(t)$ is less restricted than that used by McShane.⁷

To prove the above theorem, let $S = (x_1, x_2)$ and suppose x_3 is a third point to which we wish to extend $f(x)$. With $f(x_i)$ ($i = 1, 2$) as centers draw circles C'_i with radii $r'_i \equiv w(\|x_i, x_3\|)$, respectively. These circles are linear intervals. Suppose $C'_1 \cdot C'_2 = 0$. Then by (20) and (21) we would have

$$\begin{aligned} |f(x_1) - f(x_2)| &> w(\|x_1, x_3\|) + w(\|x_2, x_3\|) \\ &\geq w(\|x_1, x_3\| + \|x_3, x_2\|) \geq w(\|x_1, x_2\|), \end{aligned}$$

⁵ Loc. cit.

⁶ Loc. cit.

⁷ Ibid.

which contradicts the hypothesis $|f(x_1) - f(x_2)| \leq w(\|x_1, x_2\|)$. Thus $C'_1 \cdot C'_2 \neq 0$, and any point $x'_3 \in C'_1 \cdot C'_2$ serves as an extension $f(x_3)$. In order to extend $f(x)$ from an arbitrary set U on which (22) holds, to a set $U+x_0$, with $f(x_i)$ as centers, $x_i \in U$, draw circles C'_i with radii $r'_i \equiv w(\|x_i, x_0\|)$, respectively. Since each pair of the circles C'_i intersect, *Helly's*⁸ *theorem*, with $n = 1$, implies that $\prod C'_i \neq 0$. This implies that $f(x)$ can be extended from U to $U+x_0$ so as to preserve condition (22). The extension of $f(x)$ from S to the whole space now follows by transfinite induction.

An advantage of the above procedure arises from the fact that if the extension were impossible, the method would reveal it. Counterexamples exist which show that if $f(x)$ has its values in a metric space, then, in general, $f(x)$ cannot be extended so as to preserve conditions (1) and (22). For example, let x range over a two-dimensional metric space in which all the "unit spheres" are congruent squares having corresponding sides *parallel*. Let $f(x)$ have its values in the two-dimensional Euclidean space. In *Theorem 1*, let C_i ($i = 1, 2, 3$) consist of three unit squares such that $C_1 \cdot C_2 \cdot C_3 \equiv p$ consists of one and only one vertex from each square. Also choose C'_i so that $r'_i = r_i = 1$, $x'_1 = (1, 0)$, $x'_2 = (-1, 0)$, $x'_3 = (0, 1 + \epsilon)$, $\epsilon > 0$. The constant ϵ can be chosen so that condition (2) holds. Although condition (3) holds, conclusion (4) fails. Hence one cannot extend $f(x)$ from $S = (x_1, x_2, x_3)$ to $T = (x_1, x_2, x_3, p)$ so as to preserve condition (1).

The first generalization of the material in §§1 and 2 would be to the n -dimensional case, and thence to a Hilbert space. In the n -dimensional case the crux of this generalization lies in the generalization of *Theorem 1*, since that of *Helly* applies to the n -dimensional case. In the case of a Hilbert space one would also need a generalization of the theorem of *Helly*. These matters are still open questions. The author wishes to express his appreciation to Professor Max Zorn and to Dr. W. T. Puckett with whom he has had stimulating conversation concerning these topics.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES

⁸ Loc. cit.