A FAMILY OF FUNCTIONS AND ITS THEORY OF CONTACT[1]

J. F. RITT

Introduction. If $p_1, \cdots, p_n$ are fixed positive integers and $a_1, \cdots, a_n$ arbitrary constants, it is possible so to choose the a_i as to make the function

$$(1) \qquad y(x) = \prod_{i=1}^{n} (x - a_i)^{p_i}$$

and its first $p_1 + \cdots + p_n - 1$ derivatives equal to zero for any single value x_0 of x. This is accomplished by taking each a_i equal to x_0. One might say, on this basis, that *the family of polynomials* (1) *has contact of order* $p_1 + \cdots + p_n - 1$, *for every value of* x, *with* $y = 0$.

A more interesting situation is met when we allow the p_i to be any fixed positive numbers, not necessarily integral. In that case $y(x)$ may be a function of many branches, with the quotient of any two branches equal to a constant of modulus unity. For our purposes it suffices to consider the value zero of x. If no a_i is zero, each branch of $y(x)$ will be analytic at $x = 0$, with an expansion

$$c_0 + c_1 x + \cdots + c_s x^s + \cdots$$

where the c_j depend on the a_i. The question which we examine is: *What is the greatest value of* s *such that, by suitably varying the* a_i, *the coefficients* $c_0, \cdots, c_s$ *can be made to approach zero simultaneously*? Such a greatest value of s exists, and will be called, below, *the order of contact of the family* (1) *with* $y = 0$. Denoting the greatest value of s by r, we shall prove that

$$(2) \qquad r \leqq q + n - 1$$

where q is the greatest integer less than $p_1 + \cdots + p_n$. When no proper subset of the p_i has an integral sum, the equality sign holds in (2). For $n = 2$, (2) can be an inequality only when p_1 and p_2 are both integers. For $n \geqq 3$, (2) will certainly be an inequality if some integral power of $y(x)$ is a polynomial of degree not exceeding $q + n - 1$; thus the order of contact of the family

Received by the editors April 9, 1942.

[1] The problem of this note was suggested by the considerations of our paper *On the singular solutions of algebraic differential equations*, Ann. of Math. (2) vol. 37 (1936) p. 552. See also, W. C. Strodt, Trans. Amer. Math. Soc. vol. 45 (1939) p. 276.

$$y(x) = (x - a_1)^{1/2}(x - a_2)^{1/2}(x - a_3)^{1/2}$$

is two rather than three. Whether this describes all exceptional cases for $n \geqq 3$ is not decided here.

1. **The family of functions.** In what follows, the p_i in (1) will be any fixed positive numbers. A few words are necessary to make clear the meaning of the second member of (1) for given a_i. If the a_i are distinct from one another, we may take any simply connected area containing no a_i and form the product, in the area, of any selection of branches of the n functions $(x-a_i)^{p_i}$. The various products obtainable in this way are continuations of one another and are all branches of a single analytic function, which we consider the second member of (1) to represent. If two or more a_i coincide, two distinct products, as just described, need not be branches of the same analytic function. There may thus be more than one, possibly even a countable infinitude of interpretations of the second member of (1); every such analytic function will be accepted into the n-parameter family of functions (1).

Given any function y, as in (1), its values, for any x which is not an a_i, are equal in modulus; the same is true for every derivative of y.

2. **Order of contact.** Let $\mathcal{J}$ be a family of analytic functions and $f(x)$ a function[2] analytic at a point x_0. There may exist non-negative integers s which have the property that, for every $\epsilon>0$, a $g(x)$ exists in $\mathcal{J}$, with a branch analytic at x_0, such that, for this branch of $g(x)$, $g(x)-f(x)$ and its first s derivatives are less than ϵ in modulus at x_0. If such integers s exist, and if the set of them is bounded, we shall represent the greatest of them by r and shall say that $\mathcal{J}$ has *contact of order* r with $f(x)$ at x_0. If the s are unbounded, we shall say that $\mathcal{J}$ has contact of infinite order with $f(x)$ at x_0.

3. **The bound.** We examine now the functions (1). It is apparent that this family has contact of some order with $y=0$ at every point. Indeed, because the family is invariant under the addition of any constant to x, the contact with $y=0$ is the same for all values of x.

Let q be the greatest integer less than $p_1+ \cdots +p_n$. The order of contact of the family (1) with $y=0$ is not less than q. This is seen by taking all a_i equal to zero. We prove the theorem:

THEOREM. *The order of contact which the family* (1) *has with* $y=0$, *for every* x, *does not exceed* $q+n-1$.

---

[2] Not necessarily in $\mathcal{J}$.

The theorem is readily seen to be true for $n=1$; we employ induction with respect to n. We examine the theorem for $n=r>1$, assuming that it has been established for every n less than r.

We suppose the theorem false for $n=r$. Then, for $n=r$, and for certain positive numbers $p_1, \cdots, p_r$ which stay fixed during our proof, the family (1) has contact with $y=0$, for $x=0$, of order greater than $q+r-1$. Thus, if we denote the jth derivative of y by y_j,[3] we can, for every $\epsilon>0$, fix the a_i in (1) at values distinct from 0 so as to have[4]

$$|y_i(0)| < \epsilon, \qquad i = 0, 1, \cdots, q + r. \tag{3}$$

Let us show that, if ϵ is sufficiently small, each a_i, as just fixed, will have a modulus less than unity. Suppose, for instance, that for some very small ϵ, $|a_1| \geqq 1$. Then $y(x)/(x-a_1)^{p_1}$ will be very small, together with its first $q+r$ derivatives, at $x=0$. This, by the case of $n=r-1$, is impossible.

We now put

$$\alpha(x) = (x - a_1) \cdots (x - a_r);$$

$$\beta(x) = \alpha(x)\left[\frac{p_1}{x - a_1} + \cdots + \frac{p_r}{x - a_r}\right]. \tag{4}$$

We have

$$\alpha(x)y_1 - \beta(x)y = 0. \tag{5}$$

The polynomial β is of degree $r-1$. Its $(r-1)$st derivative is

$$(r - 1)!(p_1 + \cdots + p_r). \tag{6}$$

We differentiate (5) $j-1$ times, where $j \geqq 1$. Indicating derivatives of α and β by subscripts, we find that

$$\alpha y_j + [(j - 1)\alpha_1 - \beta]y_{j-1} + \left[\frac{(j - 1)(j - 2)}{2!}\alpha_2 - (j - 1)\beta_1\right]y_{j-2} + \cdots - \beta_{j-1}y = 0. \tag{7}$$

For $j \geqq r$, (7) becomes, because of the degrees of α and β,

$$\alpha y_j + \cdots + (j - 1)!\left[\frac{\alpha_r}{r!(j - r - 1)!} - \frac{\beta_{r-1}}{(r-1)!(j-r)!}\right]y_{j-r} = 0. \tag{8}$$

[3] $y_0 = y$.

[4] If y is analytic at $x=0$ when certain h of the a_i, say $a_1, \cdots, a_h$, are zero while no other a_i vanish, it must be that $p_1 + \cdots + p_h$ is integral. Thus, if $a_1, \cdots, a_h$ are changed to a common value slightly different from zero, y and any specified finite set of its derivatives will undergo only a slight change at $x=0$.

The coefficient of y_{j-r} in (8) is a constant, which, if we have regard to (6) and notice that $\alpha_r = r!$, is seen to be zero if and only if

$$p_1 + \cdots + p_r = j - r. \tag{9}$$

Let p represent $p_1 + \cdots + p_r$. If, in (1), the a_i are all multiplied by a number m, the values of $y_j(0)$ are multiplied by m^{p-j}. If $|m| > 1$, each $y_j(0)$ with $j > q$ will be multiplied by a number of modulus not greater than unity.

We consider a $y(x)$, (with definite a_i), which satisfies (3) for some very small ϵ. Let m be such that the greatest of the quantities $|ma_i|$, $i = 1, \cdots, r$, has unity for modulus. Then, by what follows (3), $|m| > 1$. Let

$$\bar{y}(x) = \prod_{i=1}^{r} (x - ma_i)^{p_i}.$$

We inspect the relation (8) as formed for $\bar{y}$. First we let $j = q + r$. In that case, (9) cannot hold. Every $|\bar{y}_i(0)|$ with $q < i \leqq q + r$ is small. Furthermore, because $|ma_i| \leqq 1$, $i = 1, \cdots, r$, there are bounds, independent of ϵ, for the values of the coefficients in (8) at $x = 0$. We infer that $|\bar{y}_q(0)|$ is small. Now, supposing that $q > 0$, let $j = q + r - 1$. We find from (8) that $|\bar{y}_{q-1}(0)|$ is small. Continuing, we find that every $|\bar{y}_i(0)|$ with $i \leqq q + r$ is small.

Let g be such that $|ma_g| = 1$. Then the function

$$\bar{y}(x)/(x - ma_g)^{p_g} \tag{10}$$

is small, together with its first $q + r$ derivatives, for $x = 0$. It is clear that we can use a single g and obtain a sequence of functions (10) which is such that the values at $x = 0$ of the kth function of the sequence and its first $q + r$ derivatives tend toward zero as k increases. By the case of $n = r - 1$, this is impossible. The theorem is proved.

4. **Attainment of bound.** We prove, for $n > 1$, the theorem:

THEOREM. *If no proper subset of the p_i has an integral sum, the family* (1) *has, for every x, contact with $y = 0$ of order precisely $q + n - 1$.*

It suffices to show that, when the p_i satisfy the hypothesis, there are values of the a_i distinct from zero such that $y_j(0) = 0$, $j = q + 1, \cdots, q + n - 1$. Such a_i being found, we can multiply them by a small m distinct from zero and obtain a function (1) which is small, for $x = 0$, together with its first $q + n - 1$ derivatives.

The existence of a_i as just described will be established if we can prove that there are numbers $b_1, \cdots, b_n$, distinct from zero, such that the function

$$z = \prod_{i=1}^{n} (1 + b_i x)^{p_i}$$

has derivatives, from the $(q+1)$st to the $(q+n-1)$st inclusive which vanish for $x=0$. The $n-1$ derivatives in question, which we represent by $Z_{q+1}, \cdots, Z_{q+n-1}$, are homogeneous polynomials in the n letters b_i. When the Z_q are equated to zero, they determine a nonvacuous algebraic manifold each of whose essential irreducible components is of dimension not less than unity.[5] Thus there is at least one set of numbers $b_1, \cdots, b_n$ which annul the Z_j and are not all zero. We assume in what follows that there is such a set in which the b_i are not all distinct from zero, and prove that some proper subset of the p_i has an integral sum.

We may now work under the assumption that, for some integer t with $0<t<n$, there exist numbers $c_1, \cdots, c_t$, all distinct from zero, such that the function

$$u = \prod_{i=1}^{t} (1 + c_i x)^{p_i}$$

has derivatives from the $(q+1)$st to the $(q+n-1)$st inclusive which vanish for $x=0$. If we put $d_i = -1/c_i$, we find that the function

$$v = \prod_{i=1}^{t} (x - d_i)^{p_i} \tag{11}$$

has derivatives from order $q+1$ through order $q+n-1$ which vanish for $x=0$. For the derivatives v_j of v, there exists a relation, analogous to (7), which expresses each v_j in terms of the derivatives which precede it if $j \leqq t$, and in terms of the t derivatives which precede it if $j>t$. In this relation, the coefficient of v_j is $(-1)^t d_1 \cdots d_t$ when $x=0$. Thus, as $v_{q+1}, \cdots, v_{q+n-1}$ vanish for $x=0$, and as they include the t derivatives which precede v_{q+n}, v_{q+n} and, then, all the derivatives which follow it, vanish for $x=0$. In other words, v is a polynomial. Thus $p_1 + \cdots + p_t$ is integral and the theorem is proved.

When the p_i are not all integers, Z_{q+1} consists of at least two terms. It is then possible to annul Z_{q+1} with b_i which are all distinct from zero, so that, by what precedes, the order of contact is at least $q+1$. In particular, when $n=2$, the order of contact is $q+1$ except when p_1 and p_2 are both integers.

COLUMBIA UNIVERSITY

---

[5] van der Waerden, *Einführung in die algebraische Geometrie*, Berlin, 1939, §41.