BOOK REVIEWS

Algebraic topology. By Solomon Lefschetz. (American Mathematical Society Colloquium Publications, vol. 27.) New York, American Mathematical Society, 1942. 6+389 pp. $6.00.

Since the publication of Lefschetz's *Topology* (Amer. Math. Soc. Colloquium Publications, vol. 12, 1930; referred to below as (L)) three major advances have influenced algebraic topology: the development of an abstract complex independent of the geometric simplex, the Pontrjagin duality theorem for abelian topological groups, and the method of Čech for treating the homology theory of topological spaces by systems of "nerves" each of which is an abstract complex. The results of (L), very materially added to both by incorporation of subsequent published work and by new theorems of the author's, are here completely recast and unified in terms of these new techniques. A high degree of generality is postulated from the outset. The abstract point of view with its concomitant formalism permits succinct, precise presentation of definitions and proofs. Examples are sparingly given, mostly of a simple kind, which, as they do not partake of the scope of the corresponding text, should be intelligible to an elementary student. But this is primarily a book for the mature reader, in which he can find the theorems of algebraic topology welded into a logically coherent whole.

The first chapter presents the set-theoretic considerations which will underlie both the spaces to be studied later and the algebraic machinery used to study them. Topological spaces are defined as point sets in which open sets are specified subject to the three usual axioms. Mappings (that is, continuous single-valued transformations) are next defined so that their properties may be developed along with those of the space. By using topological products of open or of closed line segments, Euclidean n-space, the n-cell and the Hilbert parallelotope are introduced. A space is called *compact* if for every covering $\{U_a\}$ by open sets there exists a finite subset of $\{U_a\}$ which covers the space. This is the property often called "bicompactness." The principal consequences of the separation axioms particularly for compact spaces precede a definition of *normality* and *characteristic function* and the Tychonoff imbedding theorem, which, via metric spaces, leads to the Urysohn metrization theorems. A set is a *directed system* if there is a relation $>$ between certain pairs a,b of its elements such that $c>b$ and $b>a$ implies $c>a$, and for every pair a,b of elements there is an element c such that $c>a$ and $c>b$. Using these, inverse mapping systems of Hausdorff spaces and their limit spaces are defined. The chapter ends with a definition of *homotopy*, *deformation* and *retraction*.

Abelian topological groups occupy the next chapter. The standard decomposition of a group with a finite number of generators is given. The chapter as a whole leads up to duality theorems. The first is that of Pontrjagin-van Kampen which is stated for locally compact groups G and H one of which is the character group of the other, and is proved in case G is compact, H discrete (except that reference to the literature is given for the step $G \ni g \neq 0$ implies the existence of $h \in H$ such that $gh \neq 0$). Application is made to dual directed systems of groups. If a set Ω of continuous homomorphisms α of a group G into itself forms a field (taken with discrete topology) in the sense that $\alpha g + \alpha' g = (\alpha + \alpha')g$ and $(\alpha\alpha')g = \alpha(\alpha' g)$, then G is called a *vector space* and has *linear* topology if every neighborhood of the identity of G is a sum of subspaces. If G is finite-dimensional, linear reduces to discrete topology. When a *field character* of G is defined as a homomorphism $G \rightarrow \Omega$, the field characters form a vector space H called the *character space* of G. In case G and H have linear topology and are restricted by a condition analogous to compactness, a second duality theorem, analogous to the first, is proved for them.

In Chapter III, Lefschetz defines as a *complex* an abstract system based on the "abstract complex" of Tucker except that the use of negative dimensions here permits the association with each complex X of a symmetric dual complex X^* which carries its cohomology theory. Geometric entities like *polyhedral complexes* (made up of bounded convex regions) and *Euclidean complexes* (made up of simplexes) can then be treated as special cases. Homology and cohomology theory with general coefficient groups G (in case G is a discrete field the homology groups are vector spaces) and product theory are developed abstractly both for finite and for infinite complexes having local finiteness properties. It is shown, following Steenrod, that two finite complexes have the same homology groups over every G if they have the same groups for either of the *universal* coefficient groups: G the integers, or G the reals mod 1. If G is a field of characteristic π, the Betti numbers over G are the same as those mod π. Using the group duality of Chapter II gives (1) the duality of the pth homology and cohomology groups, and (2) the Alexander duality of the pth cohomology group of a closed X and the $(p-1)$st homology group of $Y-X$ when the pth and $(p-1)$st homology groups of Y are the identity. After discussion of *product* and *join* of complexes, subdivision is treated by means of the *chain-mapping* induced by the transformation of one complex to another; intersections are defined as chain mappings of the product $X \times X^*$ to X or, equivalently, of $X^* \times X^*$ to X^*. In terms of them the homology ring of Gordon and Freudenthal is set up and the theory of coincidences and fixed points is abstractly given. Combinatorial manifold is defined and the previous results are applied in this special case, the result of (1) for finite manifolds being the duality theorem of Poincaré, for infinite manifolds the theorem of (L) p. 314.

The Čech homology theory is next abstracted: the role of Alexandroff "nerves" being played by a directed set of complexes called a *net*, the role of the projection onto a covering of its refinement being taken by a chain-mapping. If whenever there is a projection it is unique, the net is called a *spectrum*, so the spectrum is the direct abstraction of the Čech situation, the net being a generalization. For a fixed dimension the homology group $\mathfrak{H}$ of a net is then the limit group of the homology groups of that dimension of the complexes of the net. The duals of these complexes yield a *conet* and cohomology groups which permit the treatment of intersections. A duality theorem like (1) is immediate; one like (2) is proved when the coefficient group is compact or a field. For a spectrum the limit groups Z of the groups of cycles and F of the groups of bounding cycles of the complexes of the spectrum may be defined leading to a *projective homology group* $Z/\overline{F}$ for each dimension. (The bar is the closure operator.) If the coefficient group is compact or a field, $\mathfrak{H} \approx Z/\overline{F}$. A *web* is a collection of objects, A, B, $\cdots$ between certain pairs of which a relation $\supset$ of "inclusion" has been established such that to each A,B there is a $C \supset A,B$, and a D with $A,B \supset D$. Thus from a web two directed sets may be formed; one *direct*, $A > B$ means $A \supset B$; one *inverse*, $A > B$ means $B \supset A$. Here the elements of the web are the subcomplexes of a complex or, more generally, the subnets of a net, and by means of the directed sets, homology theories are constructed for these webs. The principal application is to an infinite complex each of whose elements has a finite set of elements on its boundary and in which each element has a *diameter*. Then if $A,B, \cdots$ are subcomplexes made up of elements of decreasing maximum diameters one of the homology theories of the resulting web abstracts the Vietoris homology theory.

Chapters VII and VIII apply the algebraic machinery to spaces, nets being used to define homology groups and prove duality and intersection theorems for topological spaces in the manner of Čech. If the space is normal it makes no difference whether the nets are composed of nerves of open or of closed sets. Similarly webs serve for the Vietoris-Lefschetz homology theory of compact metric spaces

(L) p. 322. It is shown that the Čech and Vietoris homology groups over a discrete group G are isomorphic. Special finite coverings of a topological space by closed sets whose interiors are disjunct, called *gratings*, lead to a net which is a spectrum and whose net homology theory is the same as the Kurosch homology theory by finite closed coverings. The projective theory of this spectrum is the homology theory of Alexander-Kolmogoroff. In Chapter VIII the topological space is specialized to be a polyhedral or simplicial complex K and the covering to be by barycentric stars of the derived (that is, regularly subdivided) complexes of K. A proof of the topological invariance of the algebraic homology groups of K then quickly results from the Čech theory. The manifold, intersection and fixed point theories given earlier are specialized to this case and a discussion of the singular chains which played such a large part in (L) and of continuous chains is included. Finally differentiable complexes and group manifolds are discussed. Hopf by generalizing simplicial group manifolds (group not necessarily abelian) defined a Γ-manifold. Lefschetz further generalizes to obtain a Γ-complex and proves Hopf's theorems for it: the rational homology groups and ring of a Γ-complex are isomorphic with those of a finite product of odd-dimensional spheres; and every finite product of odd-dimensional spheres is a Γ-manifold.

In Appendix A by S. Eilenberg and Saunders MacLane are proved for infinite complexes results on universal coefficient groups similar to those of Chapter III for finite complexes. The group of group extensions of a given group by another is the algebraic tool used. In Appendix B, P. A. Smith describes his application of algebraic topology to the study of the fixed points of a periodic homeomorphism T of a topological space R into itself. His technique is first worked out for R a simplicial complex and T a simplicial homeomorphism. The algebra is that of special homology groups defined by means of T. Then by the Čech method it is extended to compact spaces, particularly those having the homology groups of the n-sphere, and there yields topological results. Some unsolved problems are described at the end of this appendix.

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Principles of mechanics. By J. L. Synge and B. A. Griffith. New York and London, McGraw-Hill, 1942. 12+514 pp. $4.50.

In their preface the authors state that mechanics stands out as a model of clarity among all the theories of deductive science, and they have succeeded very well in support of that statement in the production of this book. The notation and the arrangement are good, and