definitions, then the loop integral \( I(z, \beta) \) in (10) is developable asymptotically in the form 

\[
I(z, \beta) \sim \sum_{n=0}^{\infty} \frac{c_n}{[\log (-[\pm z])]^{[\pm \gamma] + n} \Gamma(1 - \beta - n)}.
\]

It thus appears that the presence of an algebraic singularity of \( g(w) \) presents no serious difficulty.
proofs are the algebraic translations of these proofs.
In order to avoid certain special cases it will be convenient to define

\[ T_0(n) = \begin{cases} 
1 & \text{if } n = 1, \\
0 & \text{if } n > 1.
\end{cases} \]

The function \( T_1(n) \) has already been defined as the number of ways \( n \) can be expressed as a single factor greater than 1, so we have

\[ T_1(n) = \begin{cases} 
0 & \text{if } n = 1, \\
1 & \text{if } n > 1.
\end{cases} \]

In writing an expression such as the left member of (1) it will be convenient to write it as \( \sum_{l=0}^{\infty} (-1)^{l-1}T_l(n) \) but it should be noted that all the terms in the series are zero after a certain point. This will be true of all the series which we write in this way, so we are really dealing only with finite sums.

To prove (1) we first note that from the definition of \( T_l(n) \) we have

\[ T_{l+1}(n) = \sum_{d \mid n, d \neq n} T_l(d) = \sum_{d \mid n} T_l(d) - T_l(n), \]

for \( l \geq 0 \). We let \( f(n) = \sum_{l=0}^{\infty} (-1)^l T_l(n) \) and have, by (5),

\[ \sum_{d \mid n} f(d) = \sum_{l=0}^{\infty} (-1)^l \sum_{d \mid n} T_l(d) = \sum_{l=0}^{\infty} (-1)^l \left\{ T_l(n) + T_{l+1}(n) \right\} = T_0(n). \]

Then by the Möbius inversion formula we have

\[ f(n) = \sum_{l=0}^{\infty} (-1)^l T_l(n) = \sum_{d \mid n} T_0(n/d) \mu(d) = \mu(n), \]

where we have used (3). This is equivalent to (1).

This proof has an obvious extension. Putting \( g(n) = \sum_{l=1}^{\infty} (-1)^l \cdot (l+1) T_l(n) \) we have, using (5),

\[ \sum_{d \mid n} g(d) = \sum_{l=0}^{\infty} (-1)^l (l+1) \sum_{d \mid n} T_l(d) = \sum_{l=0}^{\infty} (-1)^l (l+1) \left\{ T_l(n) + T_{l+1}(n) \right\} = \sum_{l=0}^{\infty} (-1)^l T_l(n) = f(n) = \mu(n). \]

Inverting this we obtain
\[ g(n) = \sum_{l=0}^{\infty} (-1)^l (l + 1) T_l(n) = \sum_{d|n} \mu(d) \mu(n/d) \]

\begin{align*}
(7) \quad & \begin{cases} 
1 & \text{if } n = 1, \\
(-1)^s 2^s & \text{if } n = p_1 \cdots p_i q_1 \cdots q_i, \\
0 & \text{if } n \text{ is divisible by a cube.}
\end{cases} \\
& \sum_{l=0}^{\infty} (-1)^l \frac{(l + 1)(l + 2)}{2} T_l(n) = \sum_{d|n} \mu(d) \mu(n/d) \mu(n/d) \mu(n/d) \\
(8) \quad & \begin{cases} 
1 & \text{if } n = 1, \\
(-1)^{s+t+u} 2^s & \text{if } n = p_1 \cdots p_i q_1 \cdots q_i r_1 \cdots r_u, \\
0 & \text{if } n \text{ is divisible by a fourth power.}
\end{cases}
\end{align*}

In the same way we can prove

To prove (2) we need another expression for \( T_l(n) \). From the definitions we have

\begin{align*}
(9) \quad & T_l(n) = \sum_{r_1 \cdots r_l = n; r_i > 1} 1 = \sum_{r_1 \cdots r_l = n} T_1(r_1) \cdots T_1(r_l), \\
\text{for } l \geq 1. \text{ Multiplying this by } \log n, \text{ rearranging, and using (9), we have} \\
&T_l(n) \log n = \sum_{r_1 \cdots r_l = n} T_1(r_1) \cdots T_1(r_l)(\log r_1 + \cdots + \log r_l) \\
&= l \sum_{r_1 \cdots r_l = n} T_1(r_1) \cdots T_1(r_l) \log r_l \\
&= l \sum_{r_1 | n} T_1(r_1) \log r_1 \sum_{r_1 \cdots r_{l-1} = n/r_1} T_1(r_1) \cdots T_1(r_{l-1}) \\
&= l \sum_{r_1 | n} T_1(r_1) \log r_1 \cdot T_{l-1}(n/r_1) \\
&= l \sum_{d | n} T_{l-1}(d) T_1(n/d) \log (n/d) \\
&= l \log n \sum_{d | n} T_{l-1}(d) T_1(n/d) - l \sum_{d | n} T_{l-1}(d) T_1(n/d) \log d.
\end{align*}

Reducing this by (5) and (4) we have

\[ T_l(n) \log n = l T_l(n) \log n - l \sum_{d | n} T_{l-1}(d) T_1(n/d) \log d, \]

which we can write in the form

\begin{align*}
(10) \quad & (1/(l - 1)) \sum_{d | n} T_{l-1}(d) T_1(n/d) \log d = (1/l) T_l(n) \log n.
\end{align*}
If we now let
\[ h(n) = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} T_l(n), \]
we have, by (4) and (10),
\[
\sum_{d|n} h(d) \log d = \sum_{l=1}^{\infty} \left\{ \sum_{d|n} T_l(d) T_l(n/d) \log d + T_l(n) \log n \right\}
= \sum_{l=1}^{\infty} (-1)^{l-1} \left\{ (1/l) T_l(n) \log n + (1/(l+1)) T_{l+1}(n) \log n \right\}
= T_1(n) \log n.
\]
Inverting this we obtain
\[
h(n) \log n = \sum_{d|n} T_1(n/d) \mu(d) \log n/d
= \log n \sum_{d|n} T_1(n/d) \mu(d) - \sum_{d|n} T_1(n/d) \mu(d) \log d
= \log n \left\{ \sum_{d|n} \mu(d) - \mu(n) \right\}
- \left\{ \sum_{d|n} \mu(d) \log d - \mu(n) \log n \right\}
= - \sum_{d|n} \mu(d) \log d,
\]
where we have used (4) and the fact that \( \sum_{d|n} \mu(d) = 0 \) for \( n > 1 \). Now it is easily verified that \( -\sum_{d|n} \mu(d) \log d \) takes the value \( \log p \) if \( n = p^t \) and the value 0 if \( n \) is not a prime power.\(^4\) Hence we have
\[
h(n) = \begin{cases} 
1/t & \text{if } n = p^t, \\
0 & \text{if } n \text{ is not a prime power,}
\end{cases}
\]
which is formula (2).

We can combine (6) and (12) to obtain another formula. We first note that, from (9), we have
\[
T_l(n) = \sum_{d|n} \left\{ \sum_{r_1 \cdots r_m = d} T_1(r_1) \cdots T_1(r_m) \sum_{r_{m+1} \cdots r_t = n/d} T_1(r_{m+1}) \cdots T_1(r_t) \right\}
\]
\[
(13) \quad = \sum_{d|n} T_\mu(d) T_{l-\mu}(n/d), \quad 0 \leq \mu \leq l.
\]
\(^4\) If \( n = p^m \) and \( p \) does not divide \( m \) we have \(-\sum_{d|m} \mu(d) \log d = -\sum_{d|m} \mu(d) \log \delta = -\sum_{d|m} \mu(\delta) \log \delta + \mu(p) \log (\delta p) = -\sum_{d|\delta} \mu(\delta) \log \delta - \mu(\delta) \log (\delta p) \) and this has the value 0 if \( m > 1 \), \( \log p \) if \( m = 1 \).
We let
\begin{align}
    j(n) &= \sum_{l=1}^{\infty} \sum_{\mu=1}^{l} (-1)^{l-1}(1/\mu) T_l(n) \\
    &= T_1(n) - (3/2)T_2(n) + (11/6)T_3(n) - (25/12)T_4(n) \\
    &\quad + (137/60)T_5(n) - \cdots ,
\end{align}
and have, by (13),
\begin{align}
    j(n) &= \sum_{l=1}^{\infty} \sum_{\mu=1}^{l} (-1)^{l-1}(1/\mu) \sum_{d|n} T_\mu(d)T_{l-\mu}(n/d) \\
    &= \sum_{d|n} \left\{ \sum_{l=1}^{\infty} \sum_{\mu=p^l \mu_{\geq 1}} (-1)^{l-1}(1/\mu) T_\mu(d)(-1)^rT_r(n/d) \right\} \\
    &= \sum_{d|n} h(d) j(n/d),
\end{align}
where we have used (11) and (6). Then by (6) and (12) we have
\begin{equation}
    j(n) = \sum_{p|n} (1/t) \mu(n/p^t)
\end{equation}
\(0\) if \(n\) is divisible by the squares of two distinct primes,
\[\begin{aligned}
    &\left\{ (-1)^{s+1}/(t-1) \text{ if } n = p^t q_1 \cdots q_s, t > 1, \\
    &\quad (-1)^{s-1}s \text{ if } n = q_1 \cdots q_s.
\end{aligned}\]
In quite a similar way we can prove
\begin{align}
    k(n) &= \sum_{l=2}^{\infty} (-1)^l \sum_{\nu=1 \nu \geq 1} (1/\nu \mu) T_1(n) \\
    &= T_2(n) - T_3(n) + (11/2)T_4(n) - (5/6)T_5(n) \\
    &\quad + (137/180)T_6(n) - \cdots \\
    &= \begin{cases} 
        0 & \text{if } n \text{ has more than two distinct prime factors}, \\
        2/tu & \text{if } n = p^t q^u, \\
        \sum_{\nu=1}^{t-1} (1/\nu (t-\nu)) & \text{if } n = p^t.
    \end{cases}
\end{align}
Brun notes that \(\sum_{n=1}^{m} T_1(n) = \{m\}\), \(\sum_{n=1}^{m} T_2(n) = \sum_{\mu=2}^{m} \{m/\mu\}\), \(\sum_{n=1}^{m} T_3(n) = \sum_{\mu, \nu=2}^{m} \{m/\mu \nu\}\), and so on, where \(\{x\}\) denotes the number of integers not less than 2 and not greater than \(x\). Summing (2) he then has
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\[ H(m) = \sum_{n=1}^{m} h(n) = \{m\} - (1/2) \sum_{\mu = 2}^{\infty} \{m/\mu\} \]
\[ + (1/3) \sum_{\mu, \nu = 2}^{\infty} \{m/\mu \nu\} - \cdots. \]

(17)

This formula can be used to determine \( h(n) = H(n) - H(n - 1) \) and hence, by (2), it gives a method (although impractical) of determining whether \( n \) is a prime power. A similar thing can be done with each of our formulas. For example from (14) we have

\[ J(m) = \sum_{n=1}^{m} j(n) = \{m\} - (3/2) \sum_{\mu = 2}^{\infty} \{m/\mu\} \]
\[ + (11/6) \sum_{\mu, \nu = 2}^{\infty} \{m/\mu \nu\} - \cdots. \]

Then \( j(n) = J(n) - J(n - 1) \), with the aid of (15), determines the number of prime factors of \( n \) if \( n \) is not divisible by the squares of more than one prime. As a numerical example consider \( n = 6 \). We have

\[ J(6) = \{6\} - (3/2)\{3\} + \{2\} = 1/2, \]
\[ J(5) = \{5\} - (3/2)\{5/2\} = 5/2, \]
\[ j(6) = (1/2) - (5/2) = -2, \]

and hence 6 has two distinct prime factors, each entering to the first power.

Brun also points out that from (2) we have

\[ H(m) = \sum_{n=1}^{m} h(n) = \sum_{l=1}^{\infty} (1/l) \pi(m^{1/l}). \]

The other formulas do not lead to such simple results but (16) gives

\[ K(m) = \sum_{n=1}^{m} k(n) = \sum_{\mu = 1}^{\infty} \sum_{\nu = 0}^{\infty} (1/\mu \nu) \pi((m/p^{\nu})^{1/\mu}). \]

If (10) is put into exponential form it can be stated in a more interesting way. From (10) we have

\[ [n^{T_{l}(n)}]^{1/l} = \prod_{d \mid n; d \neq n} d^{T_{l-1}(d)} \right]^{1/(l-1)}. \]

The left member can be obtained by writing \( n \) in all possible ways as a product of \( l \) factors greater than 1, multiplying them together,
and taking the $l$th root. The right member can be obtained by writing all proper divisors of $n$ in all possible ways as products of $l - 1$ factors greater than 1, multiplying them all together, and taking the $(l - 1)$th root. Our result is that these two numbers are equal. Thus for $n = 24$, $l = 3$ we have

$$[(2 \cdot 2 \cdot 6)(2 \cdot 6 \cdot 2)(6 \cdot 2 \cdot 2)(2 \cdot 3 \cdot 4)(2 \cdot 4 \cdot 3)(3 \cdot 2 \cdot 4)(3 \cdot 4 \cdot 2)$$

$$\times (4 \cdot 2 \cdot 3)(4 \cdot 3 \cdot 2)]^{1/3} = 2^9 \cdot 3^3,$$

$$[(2 \cdot 2) \times (2 \cdot 3)(3 \cdot 2) \times (2 \cdot 4)(4 \cdot 2) \times (4 \cdot 3)(3 \cdot 4) \times (6 \cdot 2)(2 \cdot 6)]^{1/2} = 2^9 \cdot 3^3.$$