ALTERNATIVE ALGEBRAS OVER AN ARBITRARY FIELD

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The results of M. Zorn concerning alternative algebras[2] are incomplete over modular fields since, in his study of alternative division algebras, Zorn restricted the characteristic of the base field to be not two or three. In this paper we present first a unified treatment of alternative division algebras which, together with Zorn's results, permits us to state that any alternative, but not associative, algebra A over an arbitrary field F is central simple (that is, simple for all scalar extensions) if and only if A is a Cayley-Dickson algebra[3] over F.

A. A. Albert in a recent paper, *Non-associative algebras* I: *Fundamental concepts and isotopy*,[4] introduced the concept of isotopy for the study of non-associative algebras. We present in the concluding section of this paper theorems concerning isotopes (with unity quantities) of alternative algebras. The reader is referred to Albert's paper, moreover, for definitions and explanations of notations which appear there and which, in the interests of brevity, have been omitted from this paper.

1. **Alternative algebras.** A distributive algebra A is called an *alternative algebra* if $ax^2=(ax)x$ and $x^2a=x(xa)$ for all elements a, x in A. That is, in terms of the so-called right and left multiplications, A is alternative if $R_{x^2}=(R_x)^2$ and $L_{x^2}=(L_x)^2$.

The following lemma, due to R. Moufang,[5] and the Theorem of Artin are well known.

LEMMA 1. *The relations* $L_xR_xR_y=R_{xy}L_x$, $R_xL_xL_y=L_{yx}R_x$, *and* $R_xL_{xy}=L_yL_xR_x$ *hold for all* a, x, y *in an alternative algebra* A.

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[1] This paper is the essential portion of the author's doctoral dissertation, written at The University of Chicago under the direction of Professor A. A. Albert, whose kindness the author gratefully acknowledges.

[2] M. Zorn, *Theorie der Alternativen Ringe*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930) pp. 123–147; *Alternativkörper und Quadratische Systeme*, ibid. vol. 9 (1933) pp. 395–402. See M. Zorn, *Alternative rings and related questions* I: *Existence of the radical*, Ann. of Math. (2) vol. 42 (1941) pp. 676–686, to verify that the Peirce decomposition in Zorn's first paper is valid for F of characteristic two.

[3] Defined over an arbitrary field by A. A. Albert in his *Quadratic forms permitting composition*, Ann. of Math. (2) vol. 43 (1942) pp. 161–177.

[4] Ann. of Math. (2) vol. 43 (1942) pp. 685–708.

[5] R. Moufang, *Zur Struktur von Alternativkörpern*, Math. Ann. vol. 110 (1934) pp. 416–430.

THEOREM OF ARTIN. *The subalgebra generated by any two elements of an alternative algebra A is associative.*

It follows from the Theorem of Artin that if x is any element of an alternative algebra A over F, and if $f(\lambda)$ is any polynomial in λ with coefficients in F, then $f(R_x)=R_{f(x)}$ and $f(L_x)=L_{f(x)}$.

Moreover, if an alternative algebra A contains a right nonsingular element x and a left nonsingular element y, then A has a unity quantity. For the identity transformation I is then a polynomial in R_x with coefficients in F, and the correspondence $x \rightarrow R_x$ is one-to-one. Thus the unity quantity of A and the inverse x^{-1} exist, and are polynomials in x, and $R_{x^{-1}}=(R_x)^{-1}$. Clearly these results hold for an alternative division algebra A.

In this paper we shall require the following lemma.

LEMMA 2. *In an alternative division algebra A, the norm of a product xy is equal to the product of the norms of x and y.*

For if $x=0$ the lemma is obvious. Otherwise L_x is nonsingular, and $L_xR_xR_y=R_{xy}L_x$ by Lemma 1. Thus $|L_x|\cdot|R_x|\cdot|R_y|=|R_{xy}|\cdot|L_x|$, and $|R_x|\cdot|R_y|=|R_{xy}|$. The conclusion follows.

Since any simple algebra A over F is central simple over its transformation center, the determination of all simple alternative algebras consists of a determination of those which are central simple. Zorn's results imply that a central simple alternative algebra over an arbitrary field is either (1) a division algebra, (2) an associative algebra, or (3) a Cayley-Dickson algebra with divisors of zero.[6] We are led directly to this theorem.

THEOREM 1. *Let A be an alternative, but not associative, central division algebra over F. Then A is an algebra of degree two and order eight over F.*

For there exists a scalar extension A_K of A such that A_K over K is not a division algebra. Since A_K is not associative, A_K is a Cayley-Dickson algebra (with divisors of zero) over K. Hence A is of degree two and order eight over F.

2. **Alternative division algebras of degree two.** We are able to make a study of alternative division algebras of degree two which is independent of Zorn's results, and (although a portion of the result is indicated in Theorem 1) we shall do so. For the proof of Theorem 2 we require the two lemmas which follow.

[6] Zorn's so-called "vector matrix algebra" is (over any field, including those of characteristic two) merely a Cayley-Dickson algebra with divisors of zero.

LEMMA 3. *Let the principal function of any element x of an alternative division algebra of degree two over F be $x^2-t(x)\cdot x+n(x)$, where $t(x)$ and $n(x)$ are in F. Then the linear transformation*

$$(1) \qquad S:\quad x\leftrightarrow xS=t(x)-x$$

is an involution of A such that $x+xS=t(x)$, and $x(xS)=(xS)x=n(x)$.

The conclusions are trivial except for showing that S is an involution of A. Clearly $S^2=I$. By Lemma 2 we have $(xy)\cdot(xy)S=n(xy)=n(x)\cdot n(y)=(x\cdot xS)\cdot n(y)$. It follows from the Theorem of Artin that $(xy)S=(xy)^{-1}\cdot(xy)\cdot(xy)S=y^{-1}x^{-1}x(xS)\cdot n(y)=n(y)\cdot y^{-1}(xS)=(yS\cdot y)y^{-1}(xS)=yS\cdot xS$. Hence S is an involution of A.

LEMMA 4. *Let B be an alternative algebra of order $2s$ over F defined as the supplementary sum $B=f_1D+\cdots+f_sD$, where $D=(1, u_2)$, $u_2^2=u_2+\alpha$, $-4\alpha\neq 1$, $f_1=1$, $f_i^2=\gamma_i\neq 0$, γ_i in F, $d\cdot f_i=f_i\cdot dS$ for all d in D, S as in (1), $(i=2,\cdots,s)$. Then, if B is a proper subalgebra of an alternative division algebra A of degree two over F, there exists an element g in A, but not in B, such that*

$$(2) \qquad g^2=\gamma\neq 0,\ \gamma \text{ in } F, \text{ and } xg=g(xS)$$

holds for all x in B.

It is evident that $f_jD\cdot f_jD\subseteq D$ and that the intersection $(f_jD\cdot f_kD)\cap D=0$ for $j\neq k$, $j, k=1, 2,\cdots, s$. In order to establish (2) it is sufficient to prove the existence of g with the trace $t(xg)=0$ for all x of B. Now if y is any element of B, then $y=d_1+f_2d_2+\cdots+f_sd_s$ with d_i in D. But $t(f_jd_i)=0$ for $j\neq 1$. Hence $t(y)=t(d_1)$.

Let v be an element of A not in B, and write $g=(\lambda_1+\lambda_2u_2)+f_2(\lambda_3+\lambda_4u_2)+\cdots+f_s(\lambda_{2s-1}+\lambda_{2s}u_2)+v$, where the λ_i are undetermined coefficients in F. Denote $t(f_jv)$ by μ_{2j-1} and $t(f_ju_2\cdot v)$ by μ_{2j}. Then the existence of g satisfying (2) is equivalent to the existence of λ_i $(i=1,\cdots,2s)$ in F such that

$$\begin{aligned}
t(g) &= 2\lambda_1+\lambda_2+\mu_1=0,\\
t(u_2g) &= \lambda_1+(1+2\alpha)\lambda_2+\mu_2=0,\\
t(f_jg) &= 2\gamma_j\lambda_{2j-1}+\gamma_j\lambda_{2j}+\mu_{2j-1}=0, && j=2,\cdots,s,\\
t(f_ju_2\cdot g) &= \gamma_j\lambda_{2j-1}-2\alpha\gamma_j\lambda_{2j}+\mu_{2j}=0, && j=2,\cdots,s.
\end{aligned}$$

But the determinant Δ of the coefficients of this system of linear equations is $\Delta=(1+4\alpha)\cdot\prod_2^s\gamma_j^2(-1-4\alpha)\neq 0$. Hence the desired solutions λ_i exist.

An algebra Q is called a *quaternion algebra* if $Q=(1, u_2, u_3, u_4)$, $u_4=u_3u_2$, $u_2^2=u_2+\alpha$, $u_3^2=\beta$, $u_2u_3=u_3(1-u_2)$, where α and $\beta\neq 0$ are in F,

$-4\alpha \neq 1$. Moreover, Q is a division algebra if and only if there exist no λ and μ in F such that $\beta = \lambda^2 + \lambda\mu - \alpha\mu^2$.

An algebra C is called a *Cayley-Dickson algebra* if $C = Q + gQ$, with elements $z = x + gy$, where x, y are quaternions, and multiplication is defined by

$$(3) \quad (x_1 + gy_1)(x_2 + gy_2) = (x_1x_2 + \gamma y_2 \cdot y_1S) + g(x_1S \cdot y_2 + x_2y_1),$$

where $g^2 = \gamma \neq 0$ in F and S is the involution (1) of Q.

THEOREM 2. *An alternative algebra A over F is a division algebra of degree two over F if and only if A is one of the following*:

(a) *a separable quadratic field, or an inseparable field of exponent two,*

(b) *a quaternion division algebra Q, or*

(c) *a Cayley-Dickson algebra $C = Q + gQ$, where Q is a division algebra and there exist no λ, μ, ρ, σ in F such that*

$$(4) \qquad \gamma = \lambda^2 + \lambda\mu - \alpha\mu^2 - \beta\rho^2 - \beta\rho\sigma + \alpha\beta\sigma^2.$$

If A is generated by less than three elements, then A is associative and is either (a) or (b). Otherwise, if the characteristic of F is not two, A contains a quaternion subalgebra Q as in (b), and an element v which is not in Q. If the characteristic of F is two, consider two cases: if A is commutative, then A is associative[7] and is either (a) or (b). If A is not commutative, there exist two noncommutative elements x, y. These generate an associative, noncommutative subalgebra (a quaternion algebra Q) of A. Also there exists an element v of A which is not in Q.

The algebra Q is a particular example, $s = 2$, of the algebra B defined in Lemma 4, $f_2 = u_3$, $\gamma_2 = \beta$. Thus there exists an element g in A, but not in $Q = B$, satisfying (2). Then A contains $A_0 = B + gB$, the elements z of A_0 being expressible uniquely as $z = x + gy$ for x, y in B. We make effective use of equation (2), Lemma 1, and the Theorem of Artin in proving that A_0 is an algebra in which multiplication is defined by (3).

For $(gy_1)(gy_2) = (gy_1)(y_2S \cdot g) = [g(y_1 \cdot y_2S)]g = [(y_2 \cdot y_1S)g]g = (y_2 \cdot y_1S)g^2 = \gamma y_2 \cdot y_1S$. Also from the fact that $y_1S = \gamma^{-1}y_1S\gamma = \gamma^{-1}y_1S \cdot g^2 = \gamma^{-1}(y_1S \cdot g)g = \gamma^{-1}(gy_1)g$, it follows that $g(x_2y_1) = (y_1S \cdot x_2S)g = [\{\gamma^{-1}(gy_1)g\}x_2S]g = \gamma^{-1}(gy_1)[g(x_2S \cdot g)] = \gamma^{-1}(gy_1)[g(gx_2)] = \gamma^{-1}(gy_1) \cdot (\gamma x_2) = (gy_1)x_2$. But then $x_1 = \gamma^{-1}(g \cdot x_1S)g$ and $g(y_2S \cdot x_1) = (gx_1) \cdot y_2S$ also. Hence $x_1(gy_2) = [\gamma^{-1}(g \cdot x_1S)g](gy_2) = \gamma^{-1}[g(x_1S \cdot g)](y_2S \cdot g) = \gamma^{-1}[g\{(x_1S \cdot g)y_2S\}]g = \gamma^{-1}[g\{(gx_1)y_2S\}]g = \gamma^{-1}[g\{g(y_2S \cdot x_1)\}]g = \gamma^{-1}(\gamma y_2S \cdot x_1)g = g(x_1S \cdot y_2)$. Hence equation (3) holds.

[7] A commutative alternative algebra over a field of characteristic not three is associative. See M. Zorn, *Theorie der Alternativen Ringe*, op. cit.

Having shown that any alternative division algebra of degree two, which is not associative, must contain the algebra A_0, we need now only to prove that A_0 is actually an alternative algebra and is not a proper subalgebra of any alternative division algebra of degree two. We may readily verify these conclusions if we refer to the matrices corresponding to the linear transformations we have used. Thus, for $z=x+gy$ in A_0,

$$R_z = \begin{pmatrix} R_x & SR_y \\ \gamma SL_y & L_x \end{pmatrix}, \qquad L_z = \begin{pmatrix} L_x & R_y \\ \gamma R_{yS} & L_{Sx} \end{pmatrix} \tag{5}$$

where the matrices $R_x, L_x, \cdots, S$ are the 4-by-4 matrices of the linear transformations induced on Q by the corresponding transformations on A_0. Computation, involving the associativity of Q, yields $R_{z^2}=(R_z)^2$ and $L_{z^2}=(L_z)^2$. Hence the Cayley-Dickson algebra $C=A_0$ is alternative. If C were a proper subalgebra of A, then A would contain a new $A_0=C+hC$ with multiplication defined by (5), x and y being in C. But computation with the matrices above reveals that if A_0 were alternative, C would be associative, which it is not.

Condition (4) follows from Lemmas 2 and 3. Let $z=x+gy$ with x, y in Q, and let S be defined by (1). Then $n(z)=z\cdot zS=(x+gy)(xS+yS\cdot gS)$ $=(x+gy)(xS-gy)=x\cdot xS-\gamma y\cdot yS=n(x)-\gamma n(y)$. Now C is a division algebra if and only if $n(z)\neq 0$ for every nonzero z in C. But $n(x)-\gamma n(y)=0$ if and only if $\gamma=n(x)[n(y)]^{-1}=n(xy^{-1})=n(v)$, the norm of v for some v in Q. Let $v=\lambda+\mu u_2+\rho u_3+\sigma u_4$. Then C is a division algebra if and only if there exist no λ, μ, ρ, σ in F such that $\gamma=n(v)=v\cdot vS=\lambda^2+\lambda\mu-\alpha\mu^2-\beta\rho^2-\beta\rho\sigma+\alpha\beta\sigma^2$.

We may combine the results of Zorn and Theorems 1 and 2 in the following manner and say: an alternative, but not associative, algebra A over an arbitrary field F is central simple if and only if A is a Cayley-Dickson algebra over F.

3. **Isotopes of alternative algebras.** Albert has proved that an algebra A with a unity quantity is associative if and only if every isotope of A with a unity quantity is associative and is equivalent to A. We consider here the corresponding problem for alternative algebras.

THEOREM 3. *Let A be an alternative algebra, and B be an isotope of A with a unity quantity. Then A has a unity quantity, and B is alternative.*

For B is equivalent to a principal isotope A_0 of A, in which products (a, x) are defined by $(a, x) = aR_x^{(0)}$ with $R_x^{(0)} = PR_{xQ}$ for nonsingular transformations P, Q. (In terms of left multiplications, $(x, a) = aL_x^{(0)}$ with $L_x^{(0)} = QL_{xP}$.) Let e be the unity quantity of A_0 and $h = eQ$. Then $I = R_e^{(0)} = PR_h$, and R_h is nonsingular. Similarly if $k = eP$, then L_k is nonsingular, and it follows from the proof in §1 that the algebra A has a unity quantity.

Now $P = R_h^{-1} = R_{h^{-1}}$. Hence $R_x^{(0)} = R_{h^{-1}} R_{xQ} = L_h R_{h^{-1}(xQ)} L_{h^{-1}}$ by Lemma 1. That is, $R_x^{(0)} = HR_{(xT)H} H^{-1}$ where $H = L_h$ and $T = QL_h^{-2}$. Hence A_0 is equivalent to an isotope A_1 of A in which products are denoted by $[a, x] = aR_x^{(1)}$ where $R_x^{(1)} = R_{xT}$. Also $[x, a] = aL_x^{(1)}$ where $L_x^{(1)} = TL_x$. Let f be the unity quantity of A_1. Then $I = L_f^{(1)} = TL_f$ and $T = L_f^{-1} = L_{f^{-1}}$. Hence $R_x^{(1)} = R_{f^{-1}x}$. Therefore $R_{[x,x]}^{(1)} = R_{xR_x^{(1)}}^{(1)} = R_{xR_{f^{-1}x}} = R_{f^{-1}\{x(f^{-1}x)\}} = R_{(f^{-1}x)^2} = R_{f^{-1}x} R_{f^{-1}x} = R_x^{(1)} R_x^{(1)}$ since A is alternative. Since B is equivalent to A_1, it follows that half of the alternative law holds in B.

But similarly B is equivalent to an isotope A_2 of A in which products are denoted by $\{x, a\} = aL_x^{(2)}$ where $L_x^{(2)} = L_{xc^{-1}}$, the element c being the unity quantity of A_2. Then $L_{\{x,x\}}^{(2)} = L_x^{(2)} L_x^{(2)}$, and the second half of the alternative law holds in B.

We complete the study of isotopy for simple alternative algebras by proving the following theorem.

THEOREM 4. *Let A be a Cayley-Dickson algebra, and B be any isotope of A with a unity quantity. Then B is equivalent to A.*

Any isotope with a unity quantity of a central simple algebra A with a unity quantity is also central simple. Therefore, any isotope B with a unity quantity of a Cayley-Dickson algebra A is also a Cayley-Dickson algebra. We shall show that B is equivalent to A.

For B is equivalent to an algebra A_1 in which products are denoted by $[a, x] = aR_x^{(1)}$ where $R_x^{(1)} = R_{f^{-1}x}$, the element f being the unity quantity of A_1. Now f, as an element in A, is contained in some quaternion subalgebra Q of A. Let x range over Q, and R be the subspace of A_1 consisting of all elements fx. Then $x \leftrightarrow fx$ is an equivalence of Q and R. For since Q contains f and is associative, $[fx, fy] = fxR_{f^{-1}fy} = f(xy)$ for all x, y in Q.

Let S be the involution of Q defined by (1), and let $z = fx$. Then the transformation

$$U: \quad z \leftrightarrow zU = f(xS)$$

is the corresponding involution of R. Now $A = Q + gQ$ as in (3). Also $A_1 = R + wR$, where $w = fg$ (and where the multiplication defining wR

is of course the multiplication in A_1). By the proof of Theorem 2, in order to show the equivalence of A and A_1 it is sufficient to show that $[w, w] = \gamma f$ and $[w, z] = [zU, w]$ for every z of R. But $[w, w] = w(f^{-1}w) = (fg)(f^{-1}fg) = fg^2 = \gamma f$, and $[w, z] = w(f^{-1}z) = (fg)(f^{-1}fx) = (fg)x = g(x \cdot fS) = (f \cdot xS)g = (f \cdot xS)(f^{-1}fg) = zU(f^{-1}w) = [zU, w]$. This proves the theorem.

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ON FIBRE SPACES. I

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In subsequent papers I propose to investigate various properties of fibre spaces.[1] The object of the fundamental Hurewicz-Steenrod definition[1] is to state a minimum[2] set of readily verifiable conditions under which the covering homotopy theorem[1] holds. An apparent defect of their definition is that it is not topologically invariant. In fact, for topological space X and metrizable non-compact space B the property "X is a fibre space over B" depends on the metric of B. The object of this note is to give a topologically invariant definition of fibre space and to show that (when B is metrizable) X is a fibre space over B in this sense if and only if B has a metric in which X is a fibre space over B in the sense of Hurewicz-Steenrod. Since the definition of fibre space is controlled by the covering homotopy theorem, an essential part of my program is to give a topologically invariant definition of uniform homotopy.

Let π be a continuous mapping of a topological space X into another topological space B. Let $\Delta = \Delta(B)$ denote the diagonal set $\sum_{b \in B}(b, b)$ of the product space $B \times B$ and let $\bar{\pi}$ denote the mapping of $X \times B$ into $B \times B$ which is induced by the mapping π according to the rule $\bar{\pi}(x, b) = (\pi(x), b)$. Thus the graph G of π is the set $\bar{\pi}^{-1}(\Delta)$, and $\bar{\pi}^{-1}(U)$ is a neighborhood of G whenever U is a neighborhood of Δ.

Any neighborhood U of Δ determines uniquely a covering of B by neighborhoods $N_U(b)$ according to the rule $b' \in N_U(b)$ when $(b, b') \in U$.

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[1] W. Hurewicz and N. E. Steenrod, Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) p. 61.

[2] How well they succeeded in this will be indicated in my next communication.