SUBSERIES OF A CONVERGENT SERIES
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In a recent paper J. D. Hill\(^1\) has discussed the mean-value of the subseries of any absolutely convergent series \(s = \sum u_n\). Simplifying his method by use of the Rademacher functions, we obtain a mapping of the subseries into the interval \(0 \leq x \leq 1\) by defining\(^2\)

\[
\phi(x) = \sum_{n=1}^{\infty} \frac{1 + R_n(x)}{2} u_n.
\]

Hill's result states that if \(\sum|u_n|\) converges, then the mean-value is given by

\[
\int_0^1 \phi(x) \, dx = s/2.
\]

In the theorem below we point out the weakest condition on the series \(\sum u_n\) for which this result persists.

**Lemma.** If (1) converges on a set of positive measure it converges almost everywhere.

Let \(D\) be the set of points on which (1) converges. Let \(x = a_1a_2a_3\ldots\) (in binary notation) be a point of \(D\). If a finite number of the \(a_i\) are changed then the new point still belongs to \(D\), for by the definition of the Rademacher functions this operation changes only a finite number of the terms of the series (1). Then \(D\) is a "homogeneous" set not of measure 0; hence it must be of measure 1.\(^3\)

**Theorem.** A necessary and sufficient condition that the series (1) converge on a set of positive measure is that the two series \(\sum u_n\) and \(\sum u_n^2\) converge. Then (1) converges almost everywhere and (2) is valid.

(i) Suppose that (1) converges on a set of positive measure. Then it must, by the lemma, converge almost everywhere. Then there exist

\(^1\) Bull. Amer. Math. Soc. vol. 48 (1942) p. 103.
\(^2\) The mapping is not 1-1 at the points \(x = k/2^n\), but this does not affect the results. For the properties of the Rademacher functions used in this paper see Kacmarcz and Steinhaus, *Le système orthogonale de M. Rademacher*, Studia Mathematica vol. 2 (1939) p. 231.
\(^3\) C. Visser, *The law of nought-or-one in the theory of probability*, Studia Mathematica vol. 7 (1938) pp. 146–147.
two points $x_0$ and $1-x_0$, symmetric in $x=1/2$, at which (1) converges. Inasmuch as $R_n(x_0)+R_n(1-x_0)=0$ we have from (1) that $\phi(x_0)+\phi(1-x_0)=u_n/2$, so that $\sum u_n$ converges. Hence $\sum u_n R_n(x)$ converges a.e., so that $\sum u_n^2$ converges.\(^4\)

(ii) Suppose now that $s=\sum u_n$ and $\sum u_n^2$ converge. Then the series of (1) converges a.e. to a function $\phi(x)$. The $R_n(x)$ are orthonormal, so that by the Riesz-Fischer theorem the series $\sum u_n R_n(x)/2$ converges in the mean to a function of $L^2$; this function must coincide a.e. with $\phi(x)-s/2$.

To establish (2) note that by the Schwarz inequality

$$\int_0^1 |\phi(x) - s/2 - \sum_{1}^{N} (u_n/2)R_n(x)|^2 \, dx \leq \left( \int_0^1 |\phi(x) - s/2 - \sum_{1}^{N} (u_n/2)R_n(x)|^2 \, dx \right)^{1/2} = o(1) \quad (N \to \infty).$$

Since $\int_0^1 R_n(x) \, dx = 0$, (2) is an immediate consequence.

We conclude with some remarks.

(i) Hill points out that if in our theorem we take $\sum u_n$ to be conditionally convergent then $D$ is of the first category\(^1\) though of measure 1.\(^5\)

(ii) Ulam notes that if $\sum |u_n|$ converges then the set of values taken on by $\phi(x)$ is a perfect set and asks what perfect sets can be obtained this way.\(^5\)

(iii) We can obtain part of the above theorem from the laws of 0 or 1 of probability.\(^6\) For (1) may be regarded as a series of independent random variables of mean-value and standard deviation $u_n/2$.

(iv) The above method furnishes a simple proof of a theorem of Steinhaus: the series (1) is $(C, 1)$ summable a.e. if and only if $\sum u_n$ is $(C, 1)$ summable and $\sum u_n^2$ converges.\(^7\)

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\(^4\) Kacmarcz and Steinhaus, op. cit. p. 234.

\(^5\) Written communication to the author.
